Coherent unit actions on braided operads and Hopf algebras

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Outline

Background

- Operad theory emerged as an efficient tool in algebraic topology in the 1960s, and was revived in the 1990s in the development of deformation theory and quantum field theory. It focuses on the universal operations performed on a given type of algebras [LV12].

Background

- Operad theory emerged as an efficient tool in algebraic topology in the 1960s, and was revived in the 1990s in the development of deformation theory and quantum field theory. It focuses on the universal operations performed on a given type of algebras [LV12].
- Free objects of various algebras (operads) $\rightsquigarrow$ Hopf algebras
  - dendriform operad
    $\rightsquigarrow$ the Foissy-Holtkamp Hopf algebra of rooted trees [Fo02]
    $\cong$ the Loday-Ronco Hopf algebra of planar binary trees [LR98]
  - tridendriform operad $\rightsquigarrow$ the Loday Hopf algebra of planar trees
  - zinbiel operad $\rightsquigarrow$ the shuffle algebra
  - commutative tridendriform operad (CTD) $\rightsquigarrow$ the quasi-shuffle algebra
  - $\mathbf{B}_\infty$-operad $\rightsquigarrow$ $\mathbf{B}_\infty$-algebra (i.e. cofree Hopf algebras) [LR06]

- Loday showed that a binary quadratic regular operad with a so-called *coherent unit action* (CUA) endows a natural Hopf algebra structure on its free objects [Lo04].
- Holtkamp further generalized this work to regular operads [Hol06].
- Similar results hold for the corresponding algebraic structures with certain commutativity [Lo07].
- A characterization and classification were achieved for binary quadratic regular operads with a CUA [EG07].

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- Initiated from the work of Connes-Kreimer on renormalization of quantum field theory, the braided construction of the Hopf algebra of planar rooted trees was given by Foissy [Fo03].

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- Initiated from the work of Connes-Kreimer on renormalization of quantum field theory, the braided construction of the Hopf algebra of planar rooted trees was given by Foissy [Fo03].
- Recently, braided structures for Rota-Baxter algebras, dendriform algebras and tridendriform algebras have attracted attention [J13, GL19, GL20]. There again the free objects can be equipped with braided Hopf algebra structures.

Purpose

- We explore a uniform approach in the context of braided operads introduced by Fiedorowicz [Fi92] to produce braided Hopf algebras.
- Related work:
  (1) L. Guo, Y. Li, *Braided Rota-Baxter algebras, quantum quasi-shuffle algebras and braided dendriform algebras*, arXiv:1901.02843, online ready at J. Algebra Appl.
  (2) L. Guo, Y. Li, *Braided dendriform and tridendriform algebras and braided Hopf algebras of planar trees*, J. Algebraic Combin. **53** (2021), 1147–1185.
  (3) L. Guo, Y. Li, *Coherent unit actions on braided operads and Hopf algebras*, Hopf Algebras, Tensor Categories and Related Topics, 137–151, Contemp. Math. **771**, Amer. Math. Soc., RI, 2021.

Notations

- $\mathbf{k}$: a ground field of characteristic 0. All linear objects are taken over $\mathbf{k}$.
- $\mathbb{S}_n$: the n-th symmetric group. $\mathbb{S}_0 = \emptyset$, $\mathbb{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ for $n \geq 1$.
- $\mathbb{B}_n$: the n-th braid group. $\mathbb{B}_0 = \emptyset$, $\mathbb{B}_n = \langle b_1, \ldots, b_{n-1} \rangle$ for $n \geq 1$.
- **braided vector space** (BVS) (V, σ): a vector space V with **Yang-Baxter operator** $\sigma \in \mathrm{End}(V^{\otimes 2})$ satisfying

$$(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma).$$

- $f : (V, \sigma) \to (V', \sigma')$ is a **homomorphism of braided vector spaces**, if $(f \otimes f)\sigma = \sigma'(f \otimes f)$.
- **BV**: the category of braided vector spaces.

Braided Hopf algebras

- BVS $(V,\sigma) \rightsquigarrow \mathbb{B}_n$-module $V^{\otimes n}$, $n \geq 1$, with b_i acting on $V^{\otimes n}$ as
$$\sigma_i := \mathrm{id}^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{id}^{\otimes(n-1-i)}, \quad 1 \leq i \leq n-1.$$
- $(V,\sigma) \rightsquigarrow (T(V),\beta) \rightsquigarrow (V^{\otimes n},\beta_{n,n})$, $n \geq 0$.
- **braided (unital) algebra** (A,μ,u,σ):
$$(\mu \otimes \mathrm{Id}_A)\sigma_2\sigma_1 = \sigma(\mathrm{Id}_A \otimes \mu), \quad (\mathrm{Id}_A \otimes \mu)\sigma_1\sigma_2 = \sigma(\mu \otimes \mathrm{Id}_A).$$
$$\sigma(u \otimes \mathrm{Id}_A) = \mathrm{Id}_A \otimes u, \quad \sigma(\mathrm{Id}_A \otimes u) = u \otimes \mathrm{Id}_A.$$
- **braided coalgebra** $(C,\Delta,\varepsilon,\sigma)$:
$$\sigma_1\sigma_2(\Delta \otimes \mathrm{Id}_C) = (\mathrm{Id}_C \otimes \Delta)\sigma,\ \sigma_2\sigma_1(\mathrm{Id}_C \otimes \Delta) = (\Delta \otimes \mathrm{Id}_C)\sigma,$$
$$(\varepsilon \otimes \mathrm{Id}_C)\sigma = \mathrm{Id}_C = (\mathrm{Id}_C \otimes \varepsilon)\sigma.$$
- **braided bialgebra** $(B,\mu,u,\Delta,\varepsilon,\sigma)$: braided algebra and coalgebra s.t.
$$\Delta\mu = (\mu \otimes \mu)\sigma_2(\Delta \otimes \Delta).$$

  braided Hopf algebra $(H,\mu,u,\Delta,\varepsilon,S,\sigma)$: braided bialgebra with antipode S.

Braided operad

Recall the notion of braided operads originally introduced by Fiedorowicz.

Definition

A **braided (algebraic) operad** is a functor $\mathcal{P} : \mathbf{BV} \to \mathbf{BV}$ equipped with natural transformations

$$\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}, \quad \eta : \mathrm{id}_{\mathbf{BV}} \to \mathcal{P}$$

s.t. $\mathcal{P}(0) = 0$ and

$$\begin{tikzcd}
(\mathcal{P} \circ \mathcal{P}) \circ \mathcal{P} \arrow[r, "="] \arrow[d, "\gamma\circ\mathrm{id}_{\mathcal{P}}"] & \mathcal{P} \circ (\mathcal{P} \circ \mathcal{P}) \arrow[r, "\mathrm{id}_{\mathcal{P}}\circ\gamma"] & \mathcal{P} \circ \mathcal{P} \arrow[d, "\gamma"] \\
\mathcal{P} \circ \mathcal{P} \arrow[rr, "\gamma"] & & \mathcal{P}
\end{tikzcd}, \quad
\begin{tikzcd}
\mathrm{id}_{\mathbf{BV}} \circ \mathcal{P} \arrow[r, "\eta\circ\mathrm{id}_{\mathcal{P}}"] \arrow[dr, "="'] & \mathcal{P} \circ \mathcal{P} \arrow[d, "\gamma"] & \mathcal{P} \circ \mathrm{id}_{\mathbf{BV}} \arrow[l, "\mathrm{id}_{\mathcal{P}}\circ\eta"'] \arrow[dl, "="] \\
& \mathcal{P} &
\end{tikzcd}$$

A **morphism of braided operads** between $(\mathcal{P}, \gamma_{\mathcal{P}}, \eta_{\mathcal{P}})$ and $(\mathcal{Q}, \gamma_{\mathcal{Q}}, \eta_{\mathcal{Q}})$ is a natural transformation $\alpha : \mathcal{P} \to \mathcal{Q}$, s.t. $\gamma_{\mathcal{Q}} \circ (\alpha, \alpha) = \alpha \circ \gamma_{\mathcal{P}}$ and $\eta_{\mathcal{Q}} = \alpha \circ \eta_{\mathcal{P}}$.

- $\mathbb{B}$**-module**: a family $M = \{M(n)\}_{n\geq 0}$ of *right* $\mathbb{B}_n$-modules $M(n)$.
  A morphism of $\mathbb{B}$-modules $f : M \to N$ is a family of homomorphisms of $\mathbb{B}_n$-modules $f_n : M(n) \to N(n)$.
- Given any $(V, \sigma) \in$ **BV**, one can define

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\mathbb{B}_n} V^{\otimes n},$$

with its Yang-Baxter operator

$$\sigma_{\mathcal{P}}((\mu, u_1, \ldots, u_i) \otimes (\nu, v_1, \ldots, v_j)) := \sum (\nu, v'_1, \ldots, v'_j) \otimes (\mu, u'_1, \ldots, u'_i),$$

for any $(\mu, u_1, \ldots, u_i) \in \mathcal{P}(i) \otimes_{\mathbb{B}_i} V^{\otimes i}$ and $(\nu, v_1, \ldots, v_j) \in \mathcal{P}(j) \otimes_{\mathbb{B}_j} V^{\otimes j}$, when

$$\beta_{ij}(u_1 \otimes \cdots \otimes u_i \underline{\otimes} v_1 \otimes \cdots \otimes v_j) := \sum v'_1 \otimes \cdots \otimes v'_j \underline{\otimes} u'_1 \otimes \cdots \otimes u'_i.$$

- A braided operad $\mathcal{P}$ consists of a $\mathbb{B}$-module $\{\mathcal{P}(n)\}_{n\geq 0}$, with the composition maps

$$\gamma(i_1,\ldots,i_k):\mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k)\to\mathcal{P}(i_1+\cdots+i_k)$$

and the unit map $\eta:\mathbf{k}\to\mathcal{P}(1)$ satisfying the associativity, the unity conditions, and also certain equivalent conditions :

$$\gamma(\mu b;\mu_1,\ldots,\mu_k)=\gamma(\mu;\mu_{b^{-1}(1)},\ldots,\mu_{b^{-1}(k)})b(i_1,\ldots,i_k),$$

$$\gamma(\mu;\mu_1c_1,\ldots,\mu_kc_k)=\gamma(\mu;\mu_1,\ldots,\mu_k)(c_1\times\cdots\times c_k).$$

- $\mathcal{P}(n)$: the space of n-ary operations.
- braided **regular** operad (BRO):

$$\mathcal{P}(n)=\mathcal{P}_n\otimes\mathbf{k}[\mathbb{B}_n],\ n\geq 0\quad\rightsquigarrow\quad\mathcal{P}(V)\simeq\bigoplus_{n\geq 0}\mathcal{P}_n\otimes V^{\otimes n}.$$

Algebras over braided operads

- **algebra over $\mathcal{P}$** or **$\mathcal{P}$-algebra** A: a BVS with a homomorphism $\theta_A : \mathcal{P}(A) \to A$ of BVS s.t.

$$\begin{tikzcd} (\mathcal{P}\circ\mathcal{P})(A) \arrow[r, "="] \arrow[d, "\gamma_A"] & \mathcal{P}(\mathcal{P}(A)) \arrow[r, "\mathcal{P}(\theta_A)"] & \mathcal{P}(A) \arrow[d, "\theta_A"] \\ \mathcal{P}(A) \arrow[rr, "\theta_A"] & & A \end{tikzcd}\ , \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "="'] & \mathcal{P}(A) \arrow[d, "\theta_A"] \\ & A \end{tikzcd}\ .$$

  A **morphism of $\mathcal{P}$-algebras** $\varphi : A \to B$ is a homomorphism of BVS s.t. $\varphi\theta_A = \theta_B\mathcal{P}(\varphi)$.

- the **free $\mathcal{P}$-algebra** generated by a BVS V is given by $(\mathcal{P}(V), \gamma_V)$ with $\eta_V : V \to \mathcal{P}(V)$, s.t.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & A \\ \mathcal{P}(V) \arrow[ur, dotted, "\exists!\,\tilde{f}"'] & \end{tikzcd}$$

Example

A dendriform algebra D is a vector space with binary operators $\prec, \succ$ s.t.

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c),$$

$$(a \succ b) \prec c = a \succ (b \prec c),$$

$$a \succ (b \succ c) = (a \prec b + a \succ b) \succ c.$$

Intrinsically, any dendriform algebra is an algebra over the BRO $(\mathcal{P}_{\mathcal{D}}, \gamma, I)$ generated by $\mathcal{P}_{\mathcal{D},1} = \mathbf{k}I$, $\mathcal{P}_{\mathcal{D},2} = \mathbf{k} \prec \oplus \mathbf{k} \succ$, with relations

$$\gamma(\prec; \prec, I) = \gamma(\prec; I, \prec + \succ),$$

$$\gamma(\prec; \succ, I) = \gamma(\succ; I, \prec),$$

$$\gamma(\succ; I, \succ) = \gamma(\succ; \prec + \succ, I),$$

$$\gamma(\prec; I, I) = \prec,\ \gamma(\succ; I, I) = \succ,$$

$$\gamma(I; \prec) = \prec,\ \gamma(I; \succ) = \succ .$$

- A **braided dendriform algebra** defined in [GL19] is a BVS (D, σ) endowed with the dendriform algebra structure $(\prec, \succ)$ on V s.t.

$$\sigma(\mathrm{Id}_D \otimes \prec) = (\prec \otimes \mathrm{Id}_D)\sigma_2\sigma_1, \quad \sigma(\prec \otimes \mathrm{Id}_D) = (\mathrm{Id}_D \otimes \prec)\sigma_1\sigma_2,$$

$$\sigma(\mathrm{Id}_D \otimes \succ) = (\succ \otimes \mathrm{Id}_D)\sigma_2\sigma_1, \quad \sigma(\succ \otimes \mathrm{Id}_D) = (\mathrm{Id}_D \otimes \succ)\sigma_1\sigma_2.$$

- The map $\theta_D : \mathcal{P}_{\mathcal{D}}(D) \to D$ is a homomorphism of BVS.

$$\begin{aligned}\sigma(a \otimes (b \prec c)) &= \sigma(\theta_D \otimes \theta_D)((I, a) \otimes (\prec, b, c)) \\ &= (\theta_D \otimes \theta_D)\sigma_{\mathcal{P}_{\mathcal{D}}}((I, a) \otimes (\prec, b, c)) = \sum (b' \prec c') \otimes a',\end{aligned}$$

so $\sigma(\mathrm{Id}_D \otimes \prec) = (\prec \otimes \mathrm{Id}_D)\sigma_2\sigma_1$. Other conditions are similar to check. Braided dendriform algebras are interpreted as braided $\mathcal{P}_{\mathcal{D}}$-algebras.

- The free $\mathcal{P}_{\mathcal{D}}$-algebra over (V, σ) is the free braided dendriform algebra, realized as the braided analogue of the Loday-Ronco algebra of planar binary rooted trees [GL20].

Example

A tridendriform algebra T is a vector space with binary operators $\prec, \succ, *$ s.t.

$$
\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c + b * c),\\
(a \succ b) \prec c &= a \succ (b \prec c),\\
a \succ (b \succ c) &= (a \prec b + a \succ b + a * b) \succ c,\\
(a * b) \prec c &= a * (b \prec c),\\
(a \prec b) * c &= a * (b \succ c),\\
(a \succ b) * c &= a \succ (b * c),\\
(a * b) * c &= a * (b * c),
\end{aligned}
$$

Intrinsically, tridendriform algebras are algebras over the BRO $(\mathcal{P}_{\mathcal{T}}, \gamma, I)$ generated by $\mathcal{P}_{\mathcal{T},1} = \mathbf{k}I$, $\mathcal{P}_{\mathcal{T},2} = \mathbf{k} \prec \oplus \mathbf{k} \succ \oplus \mathbf{k} *$, with several relations.

- A **braided tridendriform algebra** [GL20], denoted $(T, \prec, \succ, *, \sigma)$ is a BVS (T, σ) endowed with a tridendriform algebra structure $(\prec, \succ, *)$ s.t.

$$\sigma(\mathrm{Id}_T \otimes \prec) = (\prec \otimes \mathrm{Id}_T)\sigma_2\sigma_1, \quad \sigma(\prec \otimes \mathrm{Id}_T) = (\mathrm{Id}_T \otimes \prec)\sigma_1\sigma_2,$$

$$\sigma(\mathrm{Id}_T \otimes \succ) = (\succ \otimes \mathrm{Id}_T)\sigma_2\sigma_1, \quad \sigma(\succ \otimes \mathrm{Id}_T) = (\mathrm{Id}_T \otimes \succ)\sigma_1\sigma_2,$$

$$\sigma(\mathrm{Id}_T \otimes *) = (* \otimes \mathrm{Id}_T)\sigma_2\sigma_1, \quad \sigma(* \otimes \mathrm{Id}_T) = (\mathrm{Id}_T \otimes *)\sigma_1\sigma_2.$$

- Braided tridendriform algebras are exactly braided $\mathcal{P}_{\mathcal{T}}$-algebras due to the fact that the map $\theta_T : \mathcal{P}_{\mathcal{T}}(T) \to T$ is a homomorphism of BVS.
- The free $\mathcal{P}_{\mathcal{T}}$-algebra over a BVS (V, σ) is the free braided tridendriform algebra, constructed as the braided analogue of the Loday-Ronco algebra of planar rooted trees[GL20].

Coherent unit action

- $\mathcal{P}$: braided operad with $\mathcal{P}(1) = \mathbf{k}I$, $\mathcal{I}$: a 0-ary element added to $\mathcal{P}$ s.t.

$$\mathcal{P}'(i) := \begin{cases} \mathcal{P}(i), & i \geq 1, \\ \mathbf{k}\mathcal{I}, & i = 0. \end{cases}$$

- A **unit action** on $(\mathcal{P}, \gamma, I)$ is a partial extension of γ on $\mathcal{P}'$ with

$$\gamma(i_1, \dots, i_k) : \mathcal{P}'(k) \otimes \mathcal{P}'(i_1) \otimes \cdots \otimes \mathcal{P}'(i_k) \to \mathcal{P}'(i_1 + \cdots + i_k)$$

  defined for all $i_j \geq 0$ with $j = 1, \dots, k$ and $i_1 + \cdots + i_k > 0$ if $k \geq 2$.

- For $\gamma(1, 0)$ and $\gamma(0, 1)$ in the regular case, $\exists\, \alpha, \beta : \mathcal{P}_2 \to \mathbf{k}$ s.t.

$$\gamma(\mu; I, \mathcal{I}) = \alpha(\mu)I, \quad \gamma(\mu; \mathcal{I}, I) = \beta(\mu)I, \quad \forall \mu \in \mathcal{P}_2.$$

- $A^+ := \mathbf{k} \oplus A$, called a **unitary** $\mathcal{P}$-algebra, with structure map $\theta_{A^+} : \mathcal{P}'(A^+) \to A^+$ extending θ_A by sending $\mathcal{I} \in \mathcal{P}'(0)$ to $1 \in A^+$.

Definition

For any BRO $(\mathcal{P}, \gamma, I)$ generated by operation sets $M_k \subseteq \mathcal{P}_k$, $k \geq 2$, with a unit action, suppose there are operations $\star_n \in \mathcal{P}'_n$ for all $n \geq 0$ satisfying

$$\gamma(\star_n; I, \ldots, \overset{i\text{ th}}{\mathcal{I}}, \ldots, I) = \star_{n-1},\ 1 \leq i \leq n,$$

$$\gamma(\star_2; \mathcal{I}, I) = \gamma(\star_2; I, \mathcal{I}) = I.$$

Particularly, $\star_0 = \mathcal{I}$, $\star_1 = I$ and $\alpha(\star_2) = \beta(\star_2) = 1$. Further extend γ by

$$\gamma(\star_n; \overbrace{\mathcal{I}, \ldots, \mathcal{I}}^{n\text{ times}}) = \mathcal{I}.$$

In this case, such a unit action is called **coherent** (CUA), if

$$A \boxtimes B := (A \otimes \mathbf{k}) \oplus (\mathbf{k} \otimes B) \oplus (A \otimes B)$$

is again a $\mathcal{P}$-algebra defined as below for any two $\mathcal{P}$-algebras A and B.

For any $p \in M_n$, $a_i \in A^+$ and $b_i \in B^+$, $1 \leq i \leq n$ with $n \geq 2$,

$$p(a_1 \otimes b_1, \ldots, a_n \otimes b_n)$$

$$:= \begin{cases} \sum \star_n(a'_1, \ldots, a'_n) \otimes p(b'_1, \ldots, b'_n), & \text{if at least one } b_i \in B, \\ p(a_1, \ldots, a_n) \otimes 1, & \text{if all } b_i = 1,\ p(a_1, \ldots, a_n) \text{ is defined}, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

defines the structure map $\theta_{A \boxtimes B}$ of $A \boxtimes B$ as a $\mathcal{P}$-algebra.

- $\mathcal{P}$: a BRO equipped with a CUA,
  A^+: a unitary $\mathcal{P}$-algebra. Consider $A^+ \otimes A^+ \cong (A \boxtimes A)^+$. Let
  $$\Delta : A^+ \to (A \boxtimes A)^+ \cong A^+ \otimes A^+$$
  be a coassociative linear map, s.t. $\Delta(1) = 1 \otimes 1$.
- **$\mathcal{P}$-bialgebra**: $(A^+, \theta_{A^+}, \Delta)$ with Δ a morphism of unitary $\mathcal{P}$-algebras.
  connected graded $\mathcal{P}$-Hopf algebra: $A^+ = \bigoplus_{n \geq 0} A^{(n)}$ s.t. $A^{(0)} = \mathbf{k}$,
  $$\Delta(A^{(n)}) \subseteq \sum_{i=0}^{n} A^{(i)} \otimes A^{(n-i)}, \quad n \geq 0.$$

Theorem

Let $\mathcal{P}$ be a BRO equipped with a CUA and $A^+ := \mathcal{P}(V)^+$ the free unitary $\mathcal{P}$-algebra generated by a BVS V.

The coassociative morphism of $\mathcal{P}$-algebras $\Delta_A : A^+ \to A^+ \otimes A^+$ defined by

$$\Delta_A(x) = x \otimes 1 + 1 \otimes x, \quad \forall x \in V,$$

induces a connected graded $\mathcal{P}$-bialgebra, thus a braided Hopf algebra A^+.

Example

- The dendriform operad $\mathcal{P}_{\mathcal{D}}$ has a CUA with $\star_2 := \prec + \succ$, thus $\mathcal{P}_{\mathcal{D}}(V)^+$ is a connected graded $\mathcal{P}_{\mathcal{D}}$-Hopf algebra. Similar for $\mathcal{P}_{\mathcal{T}}$.
- The braided Hopf algebra structures of free braided (tri)dendriform algebras, namely, the braided analogue of the Loday-Ronco Hopf algebra of planar (binary) rooted trees in [GL20] are recovered.

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- The braided Hopf algebra structures of free braided (tri)dendriform algebras, namely, the braided analogue of the Loday-Ronco Hopf algebra of planar (binary) rooted trees in [GL20] are recovered.

Problem

Generalize to other braided (symmetric) operads:

- *braided zinbiel operad $\rightsquigarrow$ the quantum shuffle algebra??*
- *braided CTD operad $\rightsquigarrow$ the quantum quasi-shuffle algebra??*
- *the Connes-Kremier Hopf algebra of rooted trees $\rightsquigarrow$ BHA ??*

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Thank You!