Rota's Program on Algebraic Operators and operated Hopf algebras[1]

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[1]This is the joint work with Li Guo, Tianjie Zhang and Huhu Zhang etc.

Outline

Contents

Linear operators

- Throughout the development of mathematics, linear operators satisfying operator identities have played a pivot role. Such operators can be defined for various algebraic structures.
- For the introduction, we will restrict ourselves to the associative context for the sake of historical perspective and simplicity.
- We list some of the operators below.
- From the study of algebra and Galois theory arose the algebra and group homomorphisms, satisfying the operator identity

$$P(xy) = P(x)P(y).$$

Examples of linear operators

- In the study of analysis, there are the differential operator (derivation), satisfying the Leibniz rule

$$d(xy) = d(x)y + xd(y)$$

- The integral operator, satisfying integration by parts formula

$$\int_a^x f'(t)g(t)dt = f(x)g(x) - \int_a^x f(t)g'(t)dt.$$

- In his study of probability, G. Baxter [1] introduced the Rota-Baxter operator, satisfying the operator identity

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy).$$

G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960), 731-742.

Examples of linear operators

- The Reynolds operator originated from fluid mechanics(流体动力学), defined by the operator identity

$$R(x)R(y) = R(xR(y)) + R(R(x)y) - R(R(x)R(y)).$$

This operator was treated in [4] quite recently.

T. Zhang, X. Gao and L. Guo, Reynolds algebras and their free objects from bracketed words and rooted trees, *J. Pure Appl. Alg.*, **225** (2021), 106766.

Examples of linear operators

- The notion of averaging operator was explicitly defined by Kampéde Fériet satisfying

$$P(x)P(y) = P(xP(y)) = P(P(x)y).$$

- It was already implicitly studied by O. Reynolds in 1895 in turbulence theory (湍流理论) under the disguise of a Reynolds operator, since an idempotent operator is a Reynolds operator if and only if it is an averaging operator.

J. Kampéde Fériet, L'etat actuel du probléme de la turbulaence (**I** and **II**), *La Sci. Aérienne* **3** (1934), 9-34, **4** (1935), 12-52.

Examples of linear operators

- Studies in mathematics, aerodynamics (空气动力学) and signal processing (信号处理) gave rise to the Hilbert transformation (later called modified Rota-Baxter operator) satisfying

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda xy,$$

- The modified Rota-Baxter operator is related to a Rota-Baxter operator by a linear transformation.
- Let (R, P) be a Rota-Baxter algebra of weight λ. Define

$$Q := -\lambda\, id - 2P.$$

  Then (R, Q) is a modified Rota-Baxter algebra of weight $-\lambda^2$.
- Its recent name modified Rota-Baxter operator was adapted from the modified Yang-Baxter equation in Lie algebra.

Rota's program on algebraic operators

- All these operators were known to Gian-Carlo Rota, thanks to his broad interests spanning from functional analysis, probability, lattice theory, algebra and foremost combinatorics.
- Other researchers had taken operator identities with their studies of particular interests.
- Rota looked at such identities from a more transcendental perspective, probably because of his expertise as a Professor of Philosopher as well as one of Applied Mathematics at MIT.

Rota's program on algebraic operators

- Rota's program on operator identities was formulated in his 1995 survey paper:

  > *In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra.*

- Here algebras mean associative algebras.

G.-C. Rota, Baxter operators, an introduction, in Joseph P.S.Kung, Editor, Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries, Birkhäuser, Boston, 504-512, 1995.

Rota's program on algebraic operators

- We call a linear operator algebraic if the operator satisfies an algebraic operator identity.
- Then we can call Rota's program stated above the Rota's Program on Algebraic Operators.
- All the above mentioned operators are algebraic.
- Since Rota proposed his program, further exciting applications of the linear operators that Rota noted above have been found.
- For example, Differential algebra, originated from an algebraic study of differential equation by Ritt in the 1930's, have been developed by the school of Kolchin and many others into a vast area of research with applications ranging from logic to arithmetic geometry and mechanical proof of geometric theorems.

New operators after Rota's program

- After the promotion of Rota [3], Rota-Baxter algebra experienced a remarkable renaucence this century, most notably by its applications to renormalization of quantum field theory and Stotisctic process (随机过程).
- At the same time, new linear operators have merged.
- They include differential operators with weight defined by

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y).$$

which, as a natural generalization of the differential operator, was introduced in:

L. Guo and W. Keigher, On differential Rota-Baxter algebras, *J. Pure. Appl. Algebra* **212** (2008), 522-540.

New operators after Rota's program

- Modified differential operators of weight λ satisfy the operator identity
$$d(xy) = xd(y) + d(x)y + \lambda xy.$$

- Nijenhuis operators satisfy the operator identity
$$P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$$
introduced by Carinẽna *et al.* to study quantum bi-Hamiltonian systems.

- TD operators characterize by the operator identity
$$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$$

New operators after Rota's program

- A map $\mathfrak{H}$ on a group G is called a differential operator of weight 1 (resp. -1) if

$$\mathfrak{H}(xy)=\mathfrak{H}(x)x\mathfrak{H}(y)x^{-1}\quad\Big(\text{resp. }\mathfrak{H}(xy)=x\mathfrak{H}(y)x^{-1}\mathfrak{H}(x)\Big).$$

- A map $\mathfrak{B}$ on a group G is called a Rota-Baxter operator of weight 1 (resp. -1) if

$$\mathfrak{B}(x)\mathfrak{B}(y)=\mathfrak{B}\Big(x\mathfrak{B}(x)y\mathfrak{B}(x)^{-1}\Big)$$
$$(\text{resp. }\mathfrak{B}(x)\mathfrak{B}(y)=\mathfrak{B}\Big(\mathfrak{B}(x)y\mathfrak{B}(x)^{-1}x\Big)).$$

L. Guo, H. Lang and Y. Sheng, Integration and geometrization of Rota-Baxter Lie algebras, arXiv:2009.03492.

Rota's program on algebraic operators

- Such developments further shows that operator identities are ubiquitous (普遍存在的) in mathematics and motivated us to revisit Rota's Program on Algebraic Operators:

  > *finding all possible algebraic identities that can be satisfied by a linear operator on an algebra.*

- In this process, we found that the understanding of Rota's Program on Algebraic Operators depends on two key points.
- First, we need a suitable framework to formulate precisely what is an "operator identity" (or "algebraic identity").
- Second, we need to determine key properties that characterize the classes of operator identities that are of interest to other areas of mathematics, such as those listed above.

The first key point: what is an “operator identity”

- For the first point, we note that a simplified but analogous framework has already been formulated in the 1960s and subsequently explored with great success.
- This is the study of PI-rings and PI-algebras, whose elements satisfy a set of polynomial identities.

V. Drersky and E. Fromanek, Polynomial Identity Rings, *Birkhäuser*, 2004.

C. Procesi, Rings with Polynomial Identities, *Pure Appl. Math.*, vol. 17, Marcel Dekker, Inc., New York, 1973.

The first key point: what is an “operator identity”

- With the extra action of linear operators, the suitable structure is the operated algebras.
- Namely, an operated algebra is an (associative) algebra equipped with a linear operator.
- Then the free object can be realized as the operated polynomial algebras, consisting of operated polynomials.
- Thus operator identities should be taken to be operated polynomial identities (OPIs) in the operated polynomial algebra.

The second key point: key properties of operators

- Then the second key point may be interpreted as follows:

  among all OPIs, which ones are particularly consistent with the associative algebra structure (i.e. associativity) so that they are singled out for study?

- We realize this idea by making use of two related theories: rewriting systems and Gröbner-Shirshov bases.

Contents

Operated algebras

- The concept of operated algebras originated from the work of Higgins on multi-operator group in 1956.
- Soon afterwards, Kurosh proposed the notation of algebras with (one or more) linear operators, which was forgotten almost 50 years until it was rediscovered by Guo [5].
- Guo called it an Ω-operated algebra and constructed the free Ω-operated algebra on a set.

Definition (Guo, 2009)

1. *An operated monoid (resp. operated* $\mathbf{k}$*-algebra) is a monoid (resp.* $\mathbf{k}$*-algebra) U together with a map (resp.* $\mathbf{k}$*-linear map) $P_U : U \to U$.*
2. *A morphism from an operated monoid (resp.* $\mathbf{k}$*-algebra) (U, P_U) to an operated monoid (resp.* $\mathbf{k}$*-algebra) (V, P_V) is a monoid (resp.* $\mathbf{k}$*-algebra) homomorphism $f : U \to V$ such that $f \circ P_U = P_V \circ f$.*

Free operated algebras

- Now we review the construction of free operated algebras in terms of bracketed words.
- For examples,
$$xy,\ \lfloor x\rfloor,\ \lfloor x\rfloor y,\ \lfloor\lfloor x\rfloor y\rfloor$$
are bracketed words.
- For any set Y, let $M(Y)$ the free monoid on X with identity 1, and let
$$\lfloor Y\rfloor := \{\lfloor y\rfloor \mid y\in Y\}$$

Free operated algebras

- The free operated monoid over X can be naturally constructed by the limit of a directed system

$$\{\iota_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}\}_{n=0}^{\infty}$$

  of free monoids $\mathfrak{M}_n$, where the transition morphisms ι_n will be natural embeddings.

- Let

$$\mathfrak{M}_0 = M(X) \text{ and } \mathfrak{M}_1 := M(X \cup \lfloor \mathfrak{M}_0 \rfloor).$$

- Let ι_0 be the natural embedding $\iota_0 : \mathfrak{M}_0 \hookrightarrow \mathfrak{M}_1$ from the inclusion $X \hookrightarrow X \cup \lfloor \mathfrak{M} \rfloor$.

Free operated algebras

- Assuming by induction we have defined the free monoids $\mathfrak{M}_i$ and natural embeddings $\iota_i : \mathfrak{M}_i \to \mathfrak{M}_{i+1}$ for $0 \le i \le n+1$.
- Then let

$$\mathfrak{M}_{n+2} := M(X \cup \lfloor \mathfrak{M}_{n+1} \rfloor).$$

- The injection

$$\iota_{n+1} : X \cup \lfloor \mathfrak{M}_n \rfloor \hookrightarrow X \cup \lfloor \mathfrak{M}_{n+1} \rfloor,$$

extends to an embedding of free monoids

$$\iota_{n+1} : \mathfrak{M}_{n+1} = M(X \cup \lfloor \mathfrak{M}_n \rfloor) \hookrightarrow M(X \cup \lfloor \mathfrak{M}_{n+1} \rfloor) = \mathfrak{M}_{n+2}.$$

- Finally we define the monoid

$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n = \bigcup_{n \ge 0} \mathfrak{M}_n.$$

whose elements are called bracketed words or bracketed monomials on X on X.

Free operated algebras

Theorem (Guo, 2009)

Let $i : X \hookrightarrow \mathfrak{M}(X)$ and $j : \mathfrak{M}(X) \hookrightarrow \mathbf{k}\mathfrak{M}(X)$ be the natural embeddings. Then

1. *the triple $(\mathfrak{M}(X), P := \lfloor\ \rfloor, i)$ is the free operated monoid on X;*
2. *the triple $(\mathbf{k}\mathfrak{M}(X), P, j \circ i)$ is the free operated algebra on X, where P is the linear operator induced by $\lfloor\ \rfloor$.*

- Thanks to the above construction of free operated algebras, we now can characterize precisely what an operator identity is in Rota's Program on Algebraic Operators.
- Namely, let $\phi = \phi(x_1, \ldots, x_k) \in \mathbf{k}\mathfrak{M}(X)$. We call $\phi = 0$ (or simply ϕ) an operated polynomial identity (OPI).

Contents

Rewriting systems

Definition

Let V be a vector space with a linear basis W.

1. *A term-rewriting system Π on V with respect to W is a binary relation $\Pi \subseteq W \times V$.*
2. *An element $(t, v) \in \Pi$ is called a (term-) rewriting rule of Π, denoted by $t \to v$.*

Definition

A term-rewriting system Π on V is called

1. *terminating if there is no infinite chain of one-step rewritings*

$$f_0 \to_\Pi f_1 \to_\Pi f_2 \to_\Pi \cdots \quad .$$

2. *confluent if every fork is joinable.*
3. *convergent if it is both terminating and confluent.*

$\star$-bracketed word

We list some basic notations for Gröbner-Shirshov bases.

Definition

Let X be a set, let $\star$ be a symbol not in X, and let $X^\star = X \cup \{\star\}$.

1. *By a $\star$-bracketed word on X, we mean any word in $\mathfrak{M}(X^\star)$ with exactly one occurrence of $\star$. For example, $x\star$, $\lfloor\star\rfloor$. The set of all $\star$-bracketed words on X is denoted by $\mathfrak{M}^\star(X)$.*
2. *For $q \in \mathfrak{M}^\star(X)$ and $u \in \mathfrak{M}(X)$, we define $q|_{\star\mapsto u}$ (or $q|_u$) to be the bracketed word on X obtained by replacing the symbol $\star$ in q by u. For example*
$$x\star|_y = xy,\ \lfloor\star\rfloor_x = \lfloor x\rfloor.$$
3. *For $q \in \mathfrak{M}^\star(X)$ and $s = \sum_i c_i u_i \in \mathbf{k}\mathfrak{M}(X)$, where $c_i \in \mathbf{k}$ and $u_i \in \mathfrak{M}(X)$, we define $q|_s := \sum_i c_i q|_{u_i}$.*

Monomial order

A monomial order is a well order that is compatible with all operations in the algebraic structure.

Definition

Let X be a set. A monomial order on $\mathfrak{M}(X)$ is a well order $\geq$ on $\mathfrak{M}(X)$ such that

$$u > v \Rightarrow q|_u > q|_v, \quad \text{for all } u, v \in \mathfrak{M}(X) \text{ and } q \in \mathfrak{M}^\star(X).$$

Composition

Let $\bar{f}$ be the leading bracketed word (monomial) of f.

Definition

Let $\geq$ be a monomial order on $\mathfrak{M}(X)$ and $f, g \in \mathbf{k}\mathfrak{M}(X)$ be monic.

1. *If there are $w, u, v \in \mathfrak{M}(X)$ such that $w = \bar{f}u = v\bar{g}$ with $\max\{|\bar{f}|, |\bar{g}|\} < |w| < |\bar{f}| + |\bar{g}|$, we call*

$$(f, g)_w^{u,v} := fu - vg$$

   the intersection composition of f and g with respect to w.

2. *If there are $w \in \mathfrak{M}(X)$ and $q \in \mathfrak{M}^\star(X)$ such that $w = \bar{f} = q|_{\bar{g}}$, we call*

$$(f, g)_w^q := f - q|_g$$

   the including composition of f and g with respect to w.

The above w is called an ambiguity.

An example

Consider the Rota-Baxter equation

$$\lfloor x\rfloor\lfloor y\rfloor - \lfloor x\lfloor y\rfloor\rfloor - \lfloor\lfloor x\rfloor y\rfloor = 0.$$

Then

$$w = \lfloor x\rfloor\lfloor y\rfloor\lfloor z\rfloor = \bar{f}\lfloor z\rfloor = \lfloor x\rfloor\bar{g}$$

is an ambiguity, with

$$f = \lfloor x\rfloor\lfloor y\rfloor - \lfloor x\lfloor y\rfloor\rfloor - \lfloor\lfloor x\rfloor y\rfloor, \quad g = \lfloor y\rfloor\lfloor z\rfloor - \lfloor y\lfloor z\rfloor\rfloor - \lfloor\lfloor y\rfloor z\rfloor.$$

The corresponding intersection composition is $f\lfloor z\rfloor - \lfloor x\rfloor g$.
Also

$$w = \lfloor p|_{\lfloor r\rfloor\lfloor s\rfloor}\rfloor\lfloor y\rfloor = \bar{f} = q|_{\bar{g}}$$

is another ambiguity, with $q = \lfloor p\rfloor\lfloor y\rfloor$ and

$$\begin{aligned} f =& \lfloor p|_{\lfloor r\rfloor\lfloor s\rfloor}\rfloor\lfloor y\rfloor - \lfloor p|_{\lfloor r\rfloor\lfloor s\rfloor}\lfloor y\rfloor\rfloor - \lfloor\lfloor p|_{\lfloor r\rfloor\lfloor s\rfloor}\rfloor y\rfloor, \\ g =& \lfloor r\rfloor\lfloor s\rfloor - \lfloor r\lfloor s\rfloor\rfloor - \lfloor\lfloor r\rfloor s\rfloor. \end{aligned}$$

The corresponding including composition is $f - q|_g$.

Definition

Let X be a set and $\geq$ be a monomial order on $\mathfrak{M}(X)$. Let $S \subseteq \mathbf{k}\mathfrak{M}(X)$.

1. *An element $f \in \mathbf{k}\mathfrak{M}(X)$ is called trivial modulo (S, w) if*

$$f = \sum_i c_i q_i|_{s_i} \text{ with } q_i|_{s_i} < w, \quad \text{where } c_i \in \mathbf{k}, q_i \in \mathfrak{M}^\star(X), s_i \in S.$$

2. *We call S a Gröbner-Shirshov basis in $\mathbf{k}\mathfrak{M}(X)$ with respect to $\geq$ if, for all pairs $f, g \in S$, every intersection composition of the form $(f, g)_w^{u,v}$ is trivial modulo (S, w), and every including composition of the form $(f, g)_w^q$ is trivial modulo (S, w).*

Rewriting systems induced by OPIs

Definition

Let $\geq$ be a monomial order on $\mathfrak{M}(X)$ and S be a monic subset of $\mathbf{k}\mathfrak{M}(X)$. We define a rewriting system:

$$\Pi_S^{\mathrm{ass}} := \{q|_{\bar{s}} \to q|_{R(s)} \mid q \in \mathfrak{M}^\star(X), s = \bar{s} - R(s) \in S\} \subseteq \mathfrak{M}(X) \times \mathbf{k}\mathfrak{M}(X).$$

Example

$$\Pi_{diff}^{\mathrm{ass}} := \left\{ q|_{\lfloor uv \rfloor} \to q|_{x\lfloor y \rfloor + \lfloor x \rfloor y + \lambda \lfloor x \rfloor \lfloor y \rfloor} \mid q \in \mathfrak{M}^\star(X), u, v \in \mathfrak{M}(X) \right\},$$

$$\Pi_{RB}^{\mathrm{ass}} := \left\{ q|_{\lfloor u \rfloor \lfloor v \rfloor} \to q|_{\lfloor u \lfloor v \rfloor + \lfloor u \rfloor v + \lambda uv \rfloor} \mid q \in \mathfrak{M}^\star(X), u, v \in \mathfrak{M}(X) \right\}.$$

Rewriting systems vs GSB

The following is the relationship between a Gröbner-Shirshov basis of OPIs and a convergent rewriting system of OPIs.

Theorem (Gao-Guo, [4])

Let X be a set, and let $\geq$ be a monomial order on $\mathfrak{M}(X)$. Let $S \subseteq \mathbf{k}\mathfrak{M}(X)$ be monic, and let Π_S^{ass} be the term-rewriting system from S. TFAE

1. Π_S^{ass} *is convergent.*
2. Π_S^{ass} *is confluent.*
3. $\mathrm{Id(S)} \cap \mathbf{k}\mathrm{Irr(S)} = 0$.
4. $\mathrm{Id(S)} \oplus \mathbf{k}\mathrm{Irr(S)} = \mathbf{k}\mathfrak{M}(\mathrm{X})$.
5. *S is a Gröbner-Shirshov basis in $\mathbf{k}\mathfrak{M}(X)$ with respect to $\geq$.*

Two reformulations of Rota's program

On the one hand, Rota's Program on Algebraic Operators can be interpreted in terms of rewriting systems.
Let $S \subseteq \mathbf{k}\mathfrak{M}(X)$ be a system of OPIs. We call the rewriting system S convergent if Π_S^{ass} is convergent.

Problem (Gao-Guo, [4])

(Rota's Program on Algebraic Operators via rewriting systems). *Determine all convergent systems of OPIs.*

On the other hand, Rota's Program on Algebraic Operators can also be interpreted in terms of Gröbner-Shirshov bases.

Problem (Gao-Guo, [4])

(Rota's Program on Algebraic Operators via Gröbner-Shirshov bases). *Determine all Gröbner-Shirshov bases systems of OPIs.*

Two reformulations of Rota's program

The relationship between reformulations of Rota's Program on Algebraic Operators is also studied.

Corollary (Gao-Guo,[4])

With a monomial order $\geq$ on $\mathfrak{M}(X)$, the two versions of Rota's Program on Algebraic Operators are equivalent.

Contents

Algebraic Birkhoff decomposition

- Let $\mathcal{H}$ be a connected filtered Hopf algebra, that is, $\mathcal{H}$ has a decreasing filtration $\mathcal{H}_n \subseteq \mathcal{H}$, $n \geq 0$ that is compatible with the Hopf algebra structure of $\mathcal{H}$ and $\mathcal{H}_0 = \mathbb{C}$.
- Let R be a commutative Rota-Baxter algebra (of weight $\lambda = -1$) with an idempotent Rota-Baxter operator P:

$$P(x)P(y) = P(xP(y) + P(x)y - xy) \text{ for } x, y \in R.$$

- Let $\phi : \mathcal{H} \to R$ be an algebra homomorphism.
- Theorem (Algebraic Birkhoff decomposition) Given a triple $(\mathcal{H}, R, \phi)$ as above, there is a unique decomposition of algebra homomorphisms

$$\phi = \phi_-^{\star(-1)} \star \phi_+, \begin{cases} \phi_- : \mathcal{H} \to \mathbb{C} + P(R) & \text{(counter term)}, \\ \phi_+ : \mathcal{H} \to \mathbb{C} + (id - P)(R) & \text{(renormalization)} \end{cases}$$

Algebraic Birkhoff decomposition

$$
\begin{tikzcd}
\mathcal{H} \arrow[rr, "\text{formal rules}"] \arrow[dd, "\phi"'] \arrow[ddrr, "\phi_+"] & & \text{formal expressions } (= \infty!) \\
& & \text{renormalized values} \\
R & & \mathbb{C} + (id - P)(R) \arrow[u, "\varepsilon \to 0"']
\end{tikzcd}
$$

QFT renormalization

- In quantum field theory renormalization, we take the triple $(\mathcal{H}_{\mathrm{FG}}, R_{\mathrm{FG}}, \phi_{\mathrm{FG}})$ with
- Hopf algebra $\mathcal{H}_{\mathrm{FG}}$ of Feynman graphs;
- $R_{\mathrm{FG}} = C[\varepsilon^{-1}, \varepsilon]]$ of Laurent series, with the pole part projection P;
- $\phi_{\mathrm{FG}} : \mathcal{H}_{\mathrm{FG}} \to R_{\mathrm{FG}}$ from dimensional regularized Feynman rule.
- Then Algebraic Birkhoff Decomposition gives

$$\begin{tikzcd}
\mathcal{H}_{\mathrm{FG}} \arrow[r, "\text{Feynman rules}"] \arrow[dd, "\phi_{\mathrm{FG}}"'] \arrow[ddr, "\phi_{\mathrm{FG},+}"] & \text{Feynman integrals } (= \infty!) \\
& \text{renormalized values} \\
R_{\mathrm{FG}} & \mathbb{C} + (id - P)(R_{\mathrm{FG}}) \arrow[u, "\varepsilon \to 0"']
\end{tikzcd}$$

CK Hopf algebra

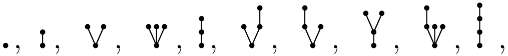
- The Connes-Kreimer Hopf algebra serves as a “baby model" of Feynmann diagrams in the algebraic approach of the renormalization in quantum field theory.
- The Connes-Kreimer Hopf algebra is built on top of rooted trees/forests, which are significant objects studied in algebra and combinatorics.
- Many other Hopf algebras has been built on rooted forests, such as Foissy-Holtkamp, Grossman-Larson and Loday-Ronco.

A. Connes and D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* **199** (1998), 203-242.

- A **rooted tree** is a finite graph, connected and without cycles, with a special vertex called the **root**.

- A **planar rooted tree** is a rooted tree with a fixed embedding into the plane.

  For example:

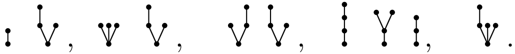


  where the root of a tree is on the bottom.

An algebraic structure on planar rooted trees

- Let $\mathcal{T}$ denote the set of planar rooted trees and $M(\mathcal{T})$ the free monoid generated by $\mathcal{T}$.

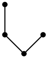
- An element in $M(\mathcal{T})$ is called a **planar rooted forest**. For example,

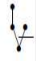
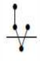
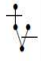
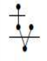


- The empty tree in $M(\mathcal{T})$ is denoted by 1.

- The triple $(\mathbf{k}M(\mathcal{T}), \mathrm{conc}, 1)$ is a unitary associative algebra.

Admissible cut

- In order to make $\mathbf{k}M(\mathcal{T})$ into a coalgebra, we now introduce the notion of cut of a tree t.
- We orient the edges of t upwards, from the root to the leaves.
- A non total cut c of a planar rooted tree t is a choice of edges of t. Deleting the chosen edges, the cut makes t into a forest denoted by $W^c(t)$.
- The cut c is called admissible if any oriented path in the tree meets at most one cut edge.
- Consider $T =$ .

cut c									total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$									

Admissible cut

- For such a cut c, the tree of $W^c(t)$ which contains the root of t is denoted by $R^c(t)$ and the concatenation product of the other trees of $W^c(t)$ is denoted by $P^c(t)$.
- We also add the total cut, which is by convention an admissible cut such that

$$R^c(t) = 1 \text{ and } P^c(t) = W^c(t) = t.$$

- Consider $T =$. As it has 3 edges, it has 2^3 non total cuts.

cut c									total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$									
$R^c(t)$					$\times$			$\times$	1
$P^c(t)$	1				$\times$			$\times$	

Coproduct

- The coproduct

$$\Delta_{\mathrm{RT}} : \mathbf{k}M(\mathcal{T}) \to \mathbf{k}M(\mathcal{T}) \otimes \mathbf{k}M(\mathcal{T})$$

is defined as the unique algebra homomorphism such that, for all $t \in \mathcal{T}$,

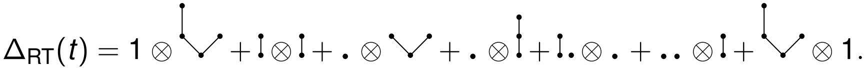
$$\Delta_{\mathrm{RT}}(t) := \sum_{c \in \mathrm{Adm}(t)} P^c(t) \otimes R^c(t).$$

An example of the coproduct

- Following the above example, we have

cut c									total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$									
$R^c(t)$					$\times$			$\times$	1
$P^c(t)$	1				$\times$			$\times$	

and

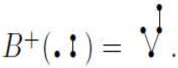


Grafting operation

- Define the grafting operation

$$B^+ : \mathbf{k}M(\mathcal{T}) \to \mathbf{k}M(\mathcal{T}), \quad t_1 \cdots t_n \mapsto B^+(t_1 \cdots t_n),$$

  where $B^+(t_1 \cdots t_n)$ is the planar rooted tree obtained by grafting the roots of $t_1, \cdots, t_n$ on a common new root.
- For example,



- The coproduct Δ_{RT} on planar rooted trees is also defined recursively on depth by $\Delta_{\mathrm{RT}}(1) := 1 \otimes 1$ and for $t = B^+(\bar{t})$,

$$\Delta_{\mathrm{RT}}(t) = \Delta_{\mathrm{RT}} B^+(\bar{t}) = t \otimes 1 + (id \otimes B^+)\Delta_{\mathrm{RT}}(\bar{t}).$$

  This equation is called the 1-cocycle condition for B^+.

Coalgebra of planar rooted trees

- Also define a linear map

$$\epsilon_{\mathrm{RT}} : \mathbf{k}M(\mathcal{T}) \to \mathbf{k}, \quad 1 \mapsto 1_{\mathbf{k}}, \quad 1 \neq f \mapsto 0.$$

- The triple $(\mathbf{k}M(\mathcal{T}), \Delta_{\mathrm{RT}}, \epsilon_{\mathrm{RT}})$ is a counitary coassociative coalgebra.

Connes-Kreimer Hopf algebra

- In summary,

Theorem (Connes-Kreimer, Foissy-Holtkamp)

The quintuple $(\mathbf{k}M(\mathcal{T}), conc, 1, \Delta_{RT}, \epsilon_{RT})$ *is a connected graded bialgebra and hence a Hopf algebra.*

A. Connes and D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* **199** (1998), 203-242.

L. Foissy, Les algèbres de Hopf desarbres enracinés I, *Bull. Sci. Math.* **126** (2002), 193-239.

R. Holtkamp, Comparison of Hopf algebras on Trees, *Arch. Math.* **80**(4) (2003), 368-383.

- Excitingly, Bruned et. al [1] used typed decorated rooted trees to give a description of a renormalisation procedure of stochastic PDEs, published at Invent. math..

Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures, *Invent. math.* **215** (2019), 1039-1156.

Operated bialgebras

Definition

1. *An operated bialgebra is a bialgebra $(H, m, u, \Delta, \varepsilon)$ which is also an operated algebra (H, P).*
2. *A cocycle bialgebra is an operated bialgebra $(H, m, u, \Delta, \varepsilon)$ that satisfies the cocycle condition:*

$$\Delta P = P \otimes 1 + (id \otimes P)\Delta.$$

If the bialgebra in a cocycle bialgebra is a Hopf algebra, then it is called a cocycle Hopf algebra.

Definition

The free cocycle bialgebra on a set X is a cocycle bialgebra

$$(H_X, m_X, u_X, \Delta_X, \varepsilon_X, P_X)$$

together with a set map $j_X : X \to H_X$ with the property that, for any cocycle bialgebra $(H, m, u, \Delta, \varepsilon, P)$ and set map $f : X \to H$ whose images are primitive (that is, $\Delta(f(x)) = f(x) \otimes 1 + 1 \otimes f(x)$), there is a unique morphism $\bar{f} : H_X \to H$ of operated bialgebras such that $\bar{f} \circ j_X = f$.

The concept of a free cocycle Hopf algebra is defined in the same way.

Free operated bialgebras

Theorem (Zhang-Gao-Guo)

1. *The septuple* $(\mathbf{k}M(\mathfrak{T}), conc, 1, \Delta_\epsilon, \epsilon_{RT}, B^+)$ *is the free cocycle bialgebra on the empty set* $\emptyset$.
2. *The Hopf algebra given by the connected bialgebra* $(\mathbf{k}M(\mathfrak{T}), conc, 1, \Delta_{RT}, \epsilon_{RT}, B^+)$ *is the free cocycle Hopf algebra on the empty set* $\emptyset$.

Free operated bialgebras

1. Let X be a set and let σ be a symbol not in the set X. Denote $\widetilde{X} := X \cup \{\sigma\}$.
2. Let $\mathcal{T}(\widetilde{X})$ (resp. $\mathcal{F}(\widetilde{X}) := M(\mathcal{T}(\widetilde{X}))$) denote the set of rooted trees (resp. forests) whose vertices are decorated by elements of $\widetilde{X}$.
3. Let $\mathcal{T}_\ell(\widetilde{X})$ (resp. $\mathcal{F}_\ell(\widetilde{X})$) denote the subset of $\mathcal{T}(\widetilde{X})$ (resp. $\mathcal{F}(\widetilde{X})$) consisting of vertex decorated trees (resp. forests) where elements of X decorate the leaves only.
4. Here are some elements in $\mathcal{T}(\widetilde{X})$:

$$\bullet_\sigma = B_\sigma^+(1),\ \ \bullet_x = x,\ \ \substack{\bullet\, \sigma \\ \bullet\, \sigma} = B_\sigma^+(B_\sigma^+(1)),\ \ \substack{\bullet\, x \\ \bullet\, \sigma} = B_\sigma^+(x),$$
$${}^{\sigma}\!\bullet\!\!\bigvee\!\!\bullet^{x}_{\sigma} = B_\sigma^+(B_\sigma^+(1)x),\ \ {}^{y}\!\bullet\!\!\bigvee\!\!\bullet^{x}_{\sigma} = B_\sigma^+(yx).$$

Free operated bialgebras

Theorem (Zhang-Gao-Guo)

Let $j_X : X \hookrightarrow \mathcal{F}_\ell(\widetilde{X}),\ x \mapsto \bullet_x$ be the natural embedding and m_{RT} the concatenation product. Then

1. *The septuple $(\mathbf{k}\mathcal{F}_\ell(\widetilde{X}),\ conc,\ 1, \Delta_\epsilon, \epsilon_{RT},\ B^+,\ j_X)$ is the free cocycle bialgebra on X.*
2. *The Hopf algebra given by the connected bialgebra $(\mathbf{k}\mathcal{F}_\ell(\widetilde{X}),\ conc,\ 1, \Delta_\epsilon, \epsilon_{RT},\ B^+,\ j_X)$ is the free cocycle Hopf algebra on X.*

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Thank you!