Rota-Baxter operators on Turaev's Hopf group (co)algebras

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Contents

Rota-Baxter algebras

Rota-Baxter algebras were intensively studied in probability and combinatorics, and more recently in mathematical physics, such as free Rota-Baxter algebras, Lie algebras, multiple zeta values, differential algebras and Connes-Kreimer renormalization theory in quantum field theory, etc.

One can refer to Li Guo's book for the detailed theory of Rota-Baxter algebras.

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MSC2020: 17B38 Yang-Baxter equations and Rota-Baxter operators

Rota-Baxter algebras

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Rota-Baxter coalgebras and bialgebras

Based on dual method, Rota-Baxter coalgebras and bialgebras were introduced. We note here the examples can be provided by the well-known Radford biproduct playing a central role in the classification of finite dimensional pointed Hopf algebras.

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Rota-Baxter family algebras

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Rota-Baxter family algebras

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Hopf group (co)algebras

The notion of (semi-)Hopf group (co)algebra was introduced by Turaev partly for reasons of homotopy field theory, which also can be used to develop certain invariants of principal π-bundles over link complements and over 3-manifolds. Examples of (crossed) π-categories can be constructed from Turaev's Hopf group-coalgebras (abbr. Hopf T-coalgebras): the category of representations of a (crossed) Hopf T-coalgebra is a (crossed) π-category. When π is abelian, a Hopf T-(co)algebra is a π-colored Hopf algebra due to Ohtsuki.

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Rota-Baxter (co)algebra

Definition 1.1

Let $\lambda \in K$. A **Rota-Baxter algebra of weight** λ *is an algebra A together with a linear map $R : A \longrightarrow A$ such that*

$$R(a)R(b) = R(aR(b)) + R(R(a)b) + \lambda R(ab)$$

for all $a, b \in A$. Such a linear operator is called a **Rota-Baxter operator of weight** λ **on** A*.*

Definition 1.2

Let $\gamma \in K$. A pair (C, Q) is called a **Rota-Baxter coalgebra of weight** γ *if C is a coalgebra and linear map $Q : C \to C$ satisfies*

$$(Q \otimes Q)\Delta = (id \otimes Q)\Delta Q + (Q \otimes id)\Delta Q + \gamma \Delta Q.$$

The map Q is called a **Rota-Baxter cooperator of weight** γ **on** C*.*

Rota-Baxter bialgebra

Definition 1.3

*Let λ, γ be elements in K and H a bialgebra (maybe without unit and counit). A triple (H, R, Q) is called a (R, Q)-**Rota-Baxter bialgebra of weight** (λ, γ) if (H, R) is a Rota-Baxter algebra of weight λ and (H, Q) is a Rota-Baxter coalgebra of weight γ.*

Rota-Baxter family algebra

Definition 1.4

A **Rota-Baxter family algebra of weight** λ *is a pair* $(A, \{R_\varphi\}_{\varphi\in\pi})$ *where* π *is a semigroup,* A *is an associative algebra and* $\{R_\varphi : A \longrightarrow A\}_{\varphi\in\pi}$ *is a family of linear operators such that*

$$R_p(a)R_q(b) = R_{pq}(R_p(a)b + aR_q(b) + \lambda ab), \forall\, a, b \in A, p, q \in \pi. \quad (1)$$

Remark 1.1

What we emphasize here is that the linear maps $\{R_\varphi\}_{\varphi\in\pi}$ are only on one linear space A.

Rota-Baxter T-algebra

Definition 2.1

Let π be a semigroup and $\lambda \in K$. A **Rota-Baxter T-algebra of weight** λ is a quadruple $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi}, \lambda)$ (abbr. (A, R)), where $\{A_\varphi\}_{\varphi\in\pi}$ is a family of vector spaces together with a family of linear maps $\{\mu_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ (write $\mu_{p,q}(a \otimes b) = a \cdot_{p,q} b$) and a family of linear maps $\{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$ such that

$$\mu_{pq,t} \circ (\mu_{p,q} \otimes id_{A_t}) = \mu_{p,qt} \circ (id_{A_p} \otimes \mu_{q,t}) \tag{2}$$

and

$$\mu_{p,q} \circ (R_p \otimes R_q) = R_{pq} \circ \mu_{p,q} \circ (R_p \otimes id_q + id_p \otimes R_q + \lambda id_p \otimes id_q). \tag{3}$$

for all $p, q, t \in \pi$.

Rota-Baxter T-algebra

If furthermore π is a monoid (i.e., a semigroup with a unit 1) and there is a linear map $\eta : K \longrightarrow A_1$ such that

$$\mu_{p,1} \circ (id_{A_p} \otimes \eta) = id_{A_p} = \mu_{1,p} \circ (\eta \otimes id_{A_p}), \tag{4}$$

then we call Rota-Baxter T-algebra (A, R) **unital.**

Rota-Baxter T-algebra

Remark 2.2

(1) If the semigroup π contains single element 1, then the Rota-Baxter T-algebra is exactly the Rota-Baxter algebra of weight λ.
(2) If $(\mathcal{A}, \mu, \eta)$ is an associative algebra, then Rota-Baxter T-algebra $(\{A_\varphi = \mathcal{A}\}, \{\mu_{p,q} = \mu\}, \{R_\varphi\}, \lambda)$ is the Rota-Baxter family algebra of weight λ.
(3) Eq.(2) $\Leftrightarrow$ $(\{A_\varphi\}, \{\mu_{p,q}\})$ (abbr. A) is a T-algebra.
(4) If we set $\{A_\varphi = \mathcal{A}\}_{\varphi\in\pi}$, then Eq.(2) is exactly the condition for S-relative.

Rota-Baxter T-algebra

Rota-Baxter T-algebras can be constructed by Rota-Baxter T-algebras in the following way.

Lemma 2.3

Let (A, R) be a Rota-Baxter T-algebra of weight λ. Define

$$\tilde{R} := \{\tilde{R}_\varphi = -\lambda id_{A_\varphi} - R_\varphi\}. \tag{5}$$

Then $(A, \tilde{R})$ is also a Rota-Baxter T-algebra of weight λ.

Characterizations

In what follows, we provide two characterizations of Rota-Baxter T-algebras. First we consider the Atkinson factorization [2] for Rota-Baxter T-algebras.

Proposition 2.4

Let π be simigroup, A a T-algebra and $R = \{R_\varphi : A_\varphi \longrightarrow A_\varphi\}$ a family of linear maps. Assume that $\lambda \in K$ have no zero divisor in $\{A_\varphi\}_{\varphi\in\pi}$ and define $\tilde{R}$ as in Eq.(5). Then (A, R) is a Rota-Baxter T-algebra of weight $\lambda \neq 0$ if and only if, for any $a \in A_p$, $b \in A_q$ and $p, q \in \pi$, there is an element $c \in A_{pq}$ such that

$$R_p(a) \cdot_{p,q} R_q(b) = R_{pq}(c), \quad \tilde{R}_p(a) \cdot_{p,q} \tilde{R}_q(b) = -\tilde{R}_{pq}(c).$$

T-quasi-idempotency

Definition 2.5

A family of linear maps $R = \{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$ is called **T-idempotent** if $R_\varphi^2 = R_\varphi$, $\forall\ \varphi \in \pi$. A family of linear maps $R = \{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$ is called **T-quasi-idempotent of weight** λ if $R_\varphi^2 = -\lambda R_\varphi, \forall\ \varphi \in \pi$.

T-quasi-idempotency

Proposition 2.6

Let π be a monoid, A a unital T-algebra and $R = \{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$ left A-linear in the sense that $R_{pq}(a \cdot_{p,q} b) = a \cdot_{p,q} R_q(b), \forall\ a \in A_p, b \in A_q, p, q \in \pi$. Then (A, R) is a Rota-Baxter T-algebra of weight λ if and only if R is T-quasi-idempotent of weight λ.

Proposition 2.7

Let (A, R) be a Rota-Baxter T-algebra of weight λ. If R is T-idempotent, then

$$(1+\lambda)R_{pq}(a \cdot_{p,q} R_q(b)) = 0, \quad (1+\lambda)R_{pq}(R_p(a) \cdot_{p,q} b) = 0,$$
$$(1+\lambda)(R_p(a) \cdot_{p,q} R_q(b) - \lambda R_{pq}(a \cdot_{p,q} b)) = 0,$$

for all $a \in A_p$, $b \in A_q$ and $p, q \in \pi$.

Examples

We will give some concrete examples from the algebras of dimensions 2,3 and 4.

Let π be a nonempty set and $\mathcal{A}$ an algebra. We denote $\mathcal{A}[\pi] = \mathcal{A} \otimes K\pi, \mathcal{A}\varphi = \mathcal{A} \otimes K\varphi, \forall \varphi \in \pi$. If, further, π is a semigroup and $\mathcal{A}$ is an algebra, then $\mathcal{A}[\pi]$ is a T-algebra with the tensor product algebra structure, i.e.,

$$\begin{aligned} \mu_{p,q} : \mathcal{A}p \otimes \mathcal{A}q &\longrightarrow \mathcal{A}(pq) \\ hp \otimes gq &\longmapsto (hg)(pq) \end{aligned}$$

for all $h, g \in \mathcal{A}$ and $p, q \in \pi$.

Examples

The following result is an analogy of [3, Proposition 9.2].

Theorem 2.8

Let π be a semigroup, $\mathcal{A}$ an algebra and $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{A}$ a linear map. For any $\varphi \in \pi$, define

$$\begin{aligned} R_\varphi : \mathcal{A}\varphi &\longrightarrow \mathcal{A}\varphi \\ h\varphi &\longmapsto \mathcal{R}(h)\varphi \end{aligned}$$

for all $h \in \mathcal{A}$. Then $(\mathcal{A}, \mathcal{R})$ is a Rota-Baxter algebra of weight λ if and only if $(\mathcal{A}[\pi], R = \{R_\varphi\})$ is a Rota-Baxter T-algebra of weight λ.

Examples

Remark 2.9

By the Theorem above, we can obtain Rota-Baxter T-algebra $(\mathcal{A}[\pi], R)$ if $(\mathcal{A}, \mathcal{R})$ is a Rota-Baxter algebra of weight λ.

Examples

Next we provide another approach to construct Rota-Baxter T-algebras.

Definition 2.10

Let $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}, \mathcal{R}')$ be two Rota-Baxter algebras of weight λ. A pair $(\mathcal{R}, \mathcal{R}')$ on $\mathcal{A}$ is called a **Rota-Baxter pair of weight** λ if it satisfies

$$\mathcal{R}(h)\mathcal{R}'(g) = \mathcal{R}'(\mathcal{R}(h)g + h\mathcal{R}'(g) + \lambda hg), \tag{6}$$

$$\mathcal{R}'(h)\mathcal{R}(g) = \mathcal{R}'(\mathcal{R}'(h)g + h\mathcal{R}(g) + \lambda hg), \tag{7}$$

for all $h, g \in \mathcal{A}$.

Examples

Remark 2.11

(1) In particular, if $(\mathcal{A}, \mathcal{R})$ is a Rota-Baxter algebra of weight λ, then $(\mathcal{R}, \mathcal{R})$ is also a Rota-Baxter pair of weight λ.
(2) If furthermore $\mathcal{A}$ is commutative, then $(\mathcal{R}, \mathcal{R}')$ is a Rota-Baxter pair if Eq.(6) or Eq.(7) holds.

Examples

The following Rota-Baxter operators on the algebra of dimension 2 were studied by de Bragança, by An and Bai, by Gubarev and by Ma, Makhlouf and Silvestrov, et al.

Example 2.12 (Dimension 2)

We consider the 2-dimensional algebra $\mathcal{A} = K\{u_1, u_2 | u_1 \cdot u_i = u_i \cdot u_1 = u_i, i = 1, 2, u_2 \cdot u_2 = u_2\}$. Then the Rota-Baxter operators on $\mathcal{A}$ of weight λ are

(a) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = 0,$

(b) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = -\lambda u_2,$

(c) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_2,$

(d) $\mathcal{R}(u_1) = -\lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2,$

(e) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_1,$

Examples

(f) $\mathcal{R}(u_1) = \lambda u_2,\ \mathcal{R}(u_2) = 0,$

(g) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2,$

(h) $\mathcal{R}(u_1) = -2\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_1,$

(i) $\mathcal{R}(u_1) = -\lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2,$

(j) $\mathcal{R}(u_1) = -\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = 0,$

(k) $\mathcal{R}(u_1) = \lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2.$

Examples

Now we list some Rota-Baxter operators on the algebras of dimensions 3 and 4 given by Ma, Makhlouf and Silvestrov.

Example 2.13 (Dimension 3)

For the 3-dimensional algebra $\mathcal{A} = K\{u_1, u_2, u_3 | u_1 \cdot u_i = u_i \cdot u_1 = u_i, i = 1, 2, 3, u_2 \cdot u_2 = u_2, u_2 \cdot u_3 = u_3 \cdot u_2 = u_3, u_3 \cdot u_3 = 0\}$, the Rota-Baxter operators of weight λ on $\mathcal{A}$ are

(a) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = 0,$

(b) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = 0,$

(c) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = -\lambda u_3,$

(d) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = 0,$

(e) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,$

(f) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = -\lambda u_3,$

(g) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,$

Examples

(h) $\mathcal{R}(u_1) = -\lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = 0,$

(i) $\mathcal{R}(u_1) = -\lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,$

(j) $\mathcal{R}(u_1) = \lambda u_2,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = 0,$

(k) $\mathcal{R}(u_1) = \lambda u_2,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = -\lambda u_3,$

(l) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2, \mathcal{R}(u_3) = 0,$

(m) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,$

(n) $\mathcal{R}(u_1) = -2\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_1, \mathcal{R}(u_3) = 0,$

(o) $\mathcal{R}(u_1) = -2\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_1, \mathcal{R}(u_3) = -\lambda u_3,$

(p) $\mathcal{R}(u_1) = -\lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = 0,$

(q) $\mathcal{R}(u_1) = -\lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,$

(r) $\mathcal{R}(u_1) = -\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = 0,$

(s) $\mathcal{R}(u_1) = -\lambda u_1 + \lambda u_2,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = -\lambda u_3,$

Examples

(t) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_1, \mathcal{R}(u_3) = 0,$

(u) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_1, \mathcal{R}(u_3) = -\lambda u_3,$

(v) $\mathcal{R}(u_1) = \lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2, \mathcal{R}(u_3) = 0,$

(w) $\mathcal{R}(u_1) = \lambda u_1 - \lambda u_2,\ \mathcal{R}(u_2) = \lambda u_1 - \lambda u_2, \mathcal{R}(u_3) = -\lambda u_3.$

Examples

Example 2.14 (Dimension 4)

For the Taft-Sweedler algebra $\mathcal{A} = T_2 = K\{u_1 = 1, u_2 = g, u_3 = x, u_4 = gx | g^2 = 1, x^2 = 0, xg = -gx\}$, its multiplication can be given by the following table.

$\cdot$	u_1	u_2	u_3	u_4
u_1	u_1	u_2	u_3	u_4
u_2	u_2	u_1	u_4	u_3
u_3	u_3	$-u_4$	0	0
u_4	u_4	$-u_3$	0	0

.

Examples

The Rota-Baxter operators on $\mathcal{A}$ of weight λ are given by

(a) $\mathcal{R}(u_1) = 0,\ \mathcal{R}(u_2) = 0, \mathcal{R}(u_3) = -\lambda u_3,\ \mathcal{R}(u_4) = -\lambda u_4,$

(b) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = 0,\ \mathcal{R}(u_4) = 0,$

(c) $\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = -\lambda u_2, \mathcal{R}(u_3) = -\lambda u_3,\ \mathcal{R}(u_4) = -\lambda u_4.$

(d) $\mathcal{R}(u_1) = 0,$
$\mathcal{R}(u_2) = -p_1u_1 + p_1u_2 - \frac{(\lambda+p_1)(\lambda+p_1+p_2)}{p_3}u_3 + \frac{(\lambda+p_1)(\lambda+p_2)}{p_3}u_4,$
$\mathcal{R}(u_3) = -p_3u_1 + p_3u_2 - (2\lambda + p_1 + p_2)u_3 + (\lambda + p_2)u_4,$
$\mathcal{R}(u_4) = -p_3u_1 + p_3u_2 - (\lambda + p_1 + p_2)u_3 + p_2u_4,$

(e) $\mathcal{R}(u_1) = -\lambda u_1,$
$\mathcal{R}(u_2) = (\lambda + p_1)u_1 + p_1u_2 - \frac{(\lambda+p_1)(\lambda+p_1+p_2)}{p_3}u_3 + \frac{(\lambda+p_1)(\lambda+p_2)}{p_3}u_4,$
$\mathcal{R}(u_3) = p_3u_1 + p_3u_2 - (2\lambda + p_1 + p_2)u_3 + (\lambda + p_2)u_4,$
$\mathcal{R}(u_4) = p_3u_1 + p_3u_2 - (\lambda + p_1 + p_2)u_3 + p_2u_4,$

Examples

(f)
$$
\begin{aligned}
\mathcal{R}(u_1) &= -\lambda u_1,\\
\mathcal{R}(u_2) &= \lambda u_1 + p_1 u_3 + \tfrac{p_1 p_2}{\lambda + p_2} u_4,\\
\mathcal{R}(u_3) &= -(\lambda + p_2) u_3 - p_2 u_4,\\
\mathcal{R}(u_4) &= (\lambda + p_2) u_3 + p_2 u_4,
\end{aligned}
$$

(g)
$$
\begin{aligned}
\mathcal{R}(u_1) &= -\lambda u_1,\\
\mathcal{R}(u_2) &= \lambda u_1 + \frac{\lambda(\lambda + p_1)}{p_2} u_3 + \frac{\lambda(\lambda + p_1)}{p_2} u_4,\\
\mathcal{R}(u_3) &= -p_2 u_1 - p_2 u_2 - (2\lambda + p_1) u_3 - (\lambda + p_1) u_4,\\
\mathcal{R}(u_4) &= p_2 u_1 + p_2 u_2 + (\lambda + p_1) u_3 + p_1 u_4,
\end{aligned}
$$

(h)
$$
\begin{aligned}
\mathcal{R}(u_1) &= \tfrac{1}{2}\lambda u_1 - \tfrac{1}{2}\lambda u_2 + p_1 u_3 + p_2 u_4,\\
\mathcal{R}(u_2) &= \tfrac{1}{2}\lambda u_1 - \tfrac{1}{2}\lambda u_2 - p_2 u_3 + p_1 u_4,\\
\mathcal{R}(u_3) &= -\tfrac{1}{2}\lambda u_3 - \tfrac{1}{2}\lambda u_4,\\
\mathcal{R}(u_4) &= -\tfrac{1}{2}\lambda u_3 - \tfrac{1}{2}\lambda u_4.
\end{aligned}
$$

Examples

On the basis of the above examples, we have

Example 2.15

Let $\mathcal{A}$ be the 2-dimensional algebra defined in Example 2.12. Then

$$(a,c),(a,d),(a,e),(a,j),(b,c),(b,d),(b,j),(c,d),(c,g),(c,j),$$
$$(d,i),(d,j),(d,k),(e,j),(f,h),(f,j),(g,j),(h,j),(i,j),(i,k),$$
$$(c,b),(d,c),(e,a),(e,b),(e,c),(e,d),(f,c),(f,d),(g,b),(g,c),$$
$$(g,d),(h,c),(h,d),(h,f),(i,c),(i,d),(j,c),(j,d),(j,f),(j,h),$$
$$(k,c),(k,d),(k,i)$$

are the Rota-Baxter pairs $(\mathcal{R},\mathcal{R}')$ on $\mathcal{A}$ of weight λ,

Examples

where (a, c) represents $(\mathcal{R}, \mathcal{R}')$, and $\mathcal{R}, \mathcal{R}'$ take the case (a) and case (c) respectively in Example 2.12, i.e.,

$$\mathcal{R}(u_1) = -\lambda u_1,\ \mathcal{R}(u_2) = 0,\ \mathcal{R}'(u_1) = -\lambda u_1,\ \mathcal{R}'(u_2) = -\lambda u_2.$$

Others are similar.

Examples

Example 2.16

Let $\mathcal{A}$ be the 3-dimensional algebra defined in Example 2.13. Then

$$(a,d),(a,g),(a,h),(a,i),(a,r),(a,t),(a,u),(b,d),(b,g),(b,h),(b,i),$$
$$(b,l),(b,r),(c,f),(c,g),(c,i),(c,r),(c,s),(c,u),(d,g),(d,h),(d,i),$$
$$(d,l),(d,r),(e,g),(e,i),(e,l),(e,m),(e,r),(e,s),(e,u),(f,g),(f,i),$$
$$(f,r),(f,s),(f,u),(g,i),(g,l),(g,m),(g,r),(g,s),(g,v),(g,w),(h,i),$$
$$(h,r),(h,v),(i,q),(i,r),(i,s),(i,v),(i,w),(j,n),(j,o),(j,r),(k,o),$$
$$(k,r),(k,s),(l,r),(m,r),(m,s),(n,o),(n,r),(o,r),(o,s),(p,r),(p,v),$$
$$(q,r),(q,s),(q,v),(q,w),(t,u),(d,b),(e,c),(f,c),(g,c),(g,e),(h,d),$$
$$(h,g),(i,c),(i,g),(j,g),(j,h),(j,i),(k,c),(k,g),(k,i),(l,b),(l,d),$$

Examples

$$(l,e),(l,g),(l,h),(l,i),(m,c),(m,e),(m,g),(m,i),(m,l),(n,d),(n,g),$$
$$(n,h),(n,i),(n,j),(o,c),(o,g),(o,i),(o,k),(p,d),(p,g),(p,h),(p,i),$$
$$(q,c),(q,g),(q,i),(r,d),(r,g),(r,h),(r,i),(r,j),(r,n),(r,o),(s,c),$$
$$(s,g),(s,i),(s,k),(s,o),(s,r),(t,a),(t,d),(t,g),(t,h),(t,i),(t,r),$$
$$(u,c),(u,f),(u,g),(u,i),(u,r),(u,s),(v,d),(v,g),(v,h),(v,i),(v,p),$$
$$(v,r),(w,c),(w,g),(w,i),(w,q),(w,r),(w,s),(w,v)$$

are the Rota-Baxter pairs $(\mathcal{R},\mathcal{R}')$ on $\mathcal{A}$ of weight λ, where $a-w$ represent the cases $(a)-(w)$ in Example 2.13.

Examples

Example 2.17

Let $\mathcal{A}$ be the 4-dimensional algebra defined in Example 2.14. Then

$$(a,c),\quad (b,c),\quad (c,a),\quad (d,c),\quad (e,c),\quad (f,c),\quad (g,c),\quad (h,c)$$

are the Rota-Baxter pairs $(\mathcal{R},\mathcal{R}')$ on $\mathcal{A}$ of weight λ, where $a-h$ represent the cases $(a)-(h)$ in Example 2.14.

Remark 2.18

In Examples 2.15-2.17, we leave out the cases of (R,R) since it can be seen as the special case of Theorem 2.8.

Examples

Theorem 2.19

Let $\pi = \{1, q\}$ be a monoid with a unit 1 and $q^2 = q$. Assume that $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}, \mathcal{R}')$ are Rota-Baxter algebras of weight λ. Define

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ h1_\pi &\longmapsto \mathcal{R}(h)1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ hq &\longmapsto \mathcal{R}'(h)q, \end{aligned}$$

then $(\mathcal{A}[\pi], \{R_\varphi\}, \lambda)$ is a Rota-Baxter T-algebra of weight λ if and only if $(\mathcal{R}, \mathcal{R}')$ is a Rota-Baxter pair.

Examples

According to Theorem 2.19, each Rota-Baxter pair determines a Rota-Baxter T-algebra of weight λ, so by Examples 2.15-2.17, we can get

Example 2.20

Let $\pi = \{1, q\}$ be a monoid with a unit 1 and $q^2 = q$.
(1) Let $\mathcal{A}$ be the 2-dimensional algebra defined in Example 2.12. Then $(\mathcal{A}[\pi], \{R_\varphi\}, \lambda)$ is a Rota-Baxter T-algebra of weight λ, where

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q. \end{aligned}$$

Other cases can be given similarly.

Examples

(2) Let $\mathcal{A}$ be the 3-dimensional algebra defined in Example 2.13. Then $(\mathcal{A}[\pi], \{R_\varphi\}, \lambda)$ is a Rota-Baxter T-algebra of weight λ, where

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto 0q. \end{aligned}$$

Other cases are similar.

Examples

(3) Let $\mathcal{A}$ be the 4-dimensional algebra defined in Example 2.14. Then $(\mathcal{A}[\pi], \{R_\varphi\}, \lambda)$ is a Rota-Baxter T-algebra of weight λ, where

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \\ u_4 1_\pi &\longmapsto -\lambda u_4 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q, \\ u_4 q &\longmapsto -\lambda u_4 q. \end{aligned}$$

Likewise we can obtain other cases.

Rota-Baxter T-algebras, (tri)dendriform T-algebras and T-algebras

The following diagram is commutative.

$$\begin{array}{ccc}
\begin{array}{c}\text{tridendriform}\\ \text{T-algebra}\end{array} & \xrightarrow{\text{Proposition } 2.29} & \text{T-algebras} \\
{\scriptstyle \text{Proposition } 2.25}\Big\Uparrow & & {\scriptstyle \text{Proposition } 2.29}\Big\Uparrow \\
\begin{array}{c}\text{Rota-Baxter}\\ \text{T-algebra}\end{array} & \xrightarrow{\text{Proposition } 2.25} & \begin{array}{c}\text{dendriform}\\ \text{T-algebra.}\end{array}
\end{array}$$

(Tri)Dendriform T-algebras

Motivated by algebraic K-theory, Loday invented the concept of a dendriform algebra. Loday and Ronco introduced the concept of a tridendriform algebra (previously also called a dendriform trialgebra) in the study of polytopes and Koszul duality. We now give the T-version.

Dendriform T-algebra

Definition 2.21

Let π be a semigroup. A **dendriform T-algebra** is a family of vector spaces $\{A_\varphi\}_{\varphi\in\pi}$ with a family of binary operations $\{\prec_{p,q}, \succ_{p,q}: A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ such that for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$,

$$(a \prec_{p,q} b) \prec_{pq,t} c = a \prec_{p,qt} (b \prec_{q,t} c + b \succ_{q,t} c),$$

$$(a \succ_{p,q} b) \prec_{pq,t} c = a \succ_{p,qt} (b \prec_{q,t} c),$$

$$(a \prec_{p,q} b + a \succ_{p,q} b) \succ_{pq,t} c = a \succ_{p,qt} (b \succ_{q,t} c).$$

For simplicity, we denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi})$ or $(A, \prec, \succ)$.

Tridendriform T-algebra

Definition 2.22

Let π be a semigroup. A **tridendriform T-algebra** is a family of vector spaces $\{A_\varphi\}_{\varphi\in\pi}$ with a family of binary operations $\{\prec_{p,q}, \succ_{p,q}, \bullet_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ such that for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$,

$$(a \prec_{p,q} b) \prec_{pq,t} c = a \prec_{p,qt} (b \prec_{q,t} c + b \succ_{q,t} c + b \bullet_{q,t} c),$$
$$(a \succ_{p,q} b) \prec_{pq,t} c = a \succ_{p,qt} (b \prec_{q,t} c),$$
$$(a \prec_{p,q} b + a \succ_{p,q} b + a \bullet_{p,q} b) \succ_{pq,t} c = a \succ_{p,qt} (b \succ_{q,t} c),$$

Tridendriform T-algebra

$$(a \succ_{p,q} b) \bullet_{pq,t} c = a \succ_{p,qt} (b \bullet_{q,t} c), \quad (8)$$

$$(a \prec_{p,q} b) \bullet_{pq,t} c = a \bullet_{p,qt} (b \succ_{q,t} c), \quad (9)$$

$$(a \bullet_{p,q} b) \prec_{pq,t} c = a \bullet_{p,qt} (b \prec_{q,t} c), \quad (10)$$

$$(a \bullet_{p,q} b) \bullet_{pq,t} c = a \bullet_{p,qt} (b \bullet_{q,t} c). \quad (11)$$

For simplicity, we denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi}, \{\bullet_{p,q}\}_{p,q\in\pi})$ or $(A, \prec, \succ, \bullet)$.

Remark

Remark 2.23

(1) If π contains a single element (i.e, π is trivial), then a (tri)dendriform T-algebra in Definition 2.21 (2.22) is exactly a (tri)dendriform algebra.
(2) (Tri)dendriform T-algebra is different from (tri)dendriform family algebra.

From tridendriform T-algebras to dendriform T-algebras

Proposition 2.24

Let $(A, \prec, \succ, \bullet)$ be a tridendriform T-algebra. Then $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec'_{p,q}\}_{p,q\in\pi}, \{\succ'_{p,q}\}_{p,q\in\pi})$ is a dendriform T-algebra, where the new operations $\{\prec'_{p,q}, \succ'_{p,q} \colon A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ are defined by

$$a \prec'_{p,q} b = a \prec_{p,q} b + a \bullet_{p,q} b, \quad a \succ'_{p,q} b = a \succ_{p,q} b$$

for $a \in A_p$, $b \in A_q$ and $p, q \in \pi$.

From Rota-Baxter T-algebras to (tri)dendriform T-algebras

Proposition 2.25

Let π *be a semigroup. (1) A Rota-Baxter T-algebra* (A, R) *induces a tridendriform T-algebra* $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi}, \{\bullet_{p,q}\}_{p,q\in\pi})$, *where*

$$a \prec_{p,q} b := a \cdot_{p,q} R_q(b),\ a \succ_{p,q} b := R_p(a) \cdot_{p,q} b$$
$$a \bullet_{p,q} b := \lambda a \cdot_{p,q} b.$$

(2) A Rota-Baxter T-algebra (A, R) *induces a dendriform T-algebra* $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi})$, *where*

$$a \prec_{p,q} b := a \cdot_{p,q} R_q(b) + \lambda a \cdot_{p,q} b, \quad a \succ_{p,q} b := R_p(a) \cdot_{p,q} b.$$

Examples

Example 2.26

By Proposition 2.25 and Example 2.20, we can obtain a variety of ways to construct (tri)dendriform algebras of $\mathcal{A}[\pi]$ with different dimensions.
(1) The new structures of tridendriform T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.12) can be defined by

$$
\begin{aligned}
& u_1 1_\pi \prec u_1 q = -\lambda u_2 q, u_1 1_\pi \succ u_1 q = -\lambda u_1 q, u_1 1_\pi \bullet u_1 q = \lambda u_1 q, \\
& u_1 1_\pi \prec u_2 q = -\lambda u_2 q, u_1 1_\pi \succ u_2 q = -\lambda u_2 q, u_1 1_\pi \bullet u_2 q = \lambda u_2 q, \\
& u_2 1_\pi \prec u_1 q = -\lambda u_2 q, u_2 1_\pi \succ u_1 q = -\lambda u_2 q, u_2 1_\pi \bullet u_1 q = \lambda u_2 q, \\
& u_2 1_\pi \prec u_2 q = -\lambda u_2 q, u_2 1_\pi \succ u_2 q = -\lambda u_2 q, u_2 1_\pi \bullet u_2 q = \lambda u_2 q, \\
& u_1 q \prec u_1 1_\pi = -\lambda u_1 q, u_1 q \succ u_1 1_\pi = -\lambda u_2 q, u_1 q \bullet u_1 1_\pi = \lambda u_1 q, \\
& u_1 q \prec u_2 1_\pi = -\lambda u_2 q, u_1 q \succ u_2 1_\pi = -\lambda u_2 q, u_1 q \bullet u_2 1_\pi = \lambda u_2 q, \\
& u_2 q \prec u_1 1_\pi = -\lambda u_2 q, u_2 q \succ u_1 1_\pi = -\lambda u_2 q, u_2 q \bullet u_1 1_\pi = \lambda u_2 q,
\end{aligned}
$$

Examples

$$u_2q \prec u_21_\pi = -\lambda u_2q, u_2q \succ u_21_\pi = -\lambda u_2q, u_2q \bullet u_21_\pi = \lambda u_2q,$$
$$u_11_\pi \prec u_11_\pi = -\lambda u_11_\pi, u_11_\pi \succ u_11_\pi = -\lambda u_11_\pi, u_11_\pi \bullet u_11_\pi = \lambda u_11_\pi,$$
$$u_11_\pi \prec u_21_\pi = -\lambda u_21_\pi, u_11_\pi \succ u_21_\pi = -\lambda u_21_\pi, u_11_\pi \bullet u_21_\pi = \lambda u_21_\pi,$$
$$u_21_\pi \prec u_11_\pi = -\lambda u_21_\pi, u_21_\pi \succ u_11_\pi = -\lambda u_21_\pi, u_21_\pi \bullet u_11_\pi = \lambda u_21_\pi,$$
$$u_21_\pi \prec u_21_\pi = -\lambda u_21_\pi, u_21_\pi \succ u_21_\pi = -\lambda u_21_\pi, u_21_\pi \bullet u_21_\pi = \lambda u_21_\pi,$$
$$u_1q \prec u_1q = -\lambda u_2q, u_1q \succ u_1q = -\lambda u_2q, u_1q \bullet u_1q = \lambda u_1q,$$
$$u_1q \prec u_2q = -\lambda u_2q, u_1q \succ u_2q = -\lambda u_2q, u_1q \bullet u_2q = \lambda u_2q,$$
$$u_2q \prec u_1q = -\lambda u_2q, u_2q \succ u_1q = -\lambda u_2q, u_2q \bullet u_1q = \lambda u_2q,$$
$$u_2q \prec u_2q = -\lambda u_2q, u_2q \succ u_2q = -\lambda u_2q, u_2q \bullet u_2q = \lambda u_2q.$$

Examples

(2) The new structures of tridendriform T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.14) can be defined by

$$u_1 1_\pi \prec u_1 q = -\lambda u_1 q,\ u_1 1_\pi \succ u_1 q = -\lambda u_1 q,$$
$$u_1 1_\pi \bullet u_1 q = \lambda u_1 q,\ u_1 1_\pi \prec u_2 q = -\lambda u_2 q,$$
$$u_1 1_\pi \succ u_2 q = -\lambda u_2 q,\ u_1 1_\pi \bullet u_2 q = \lambda u_2 q,$$
$$u_1 1_\pi \prec u_3 q = -\lambda u_3 q,\ u_1 1_\pi \succ u_3 q = -\lambda u_3 q,$$
$$u_1 1_\pi \bullet u_3 q = \lambda u_3 q,\ u_1 1_\pi \prec u_4 q = -\lambda u_4 q,$$
$$u_1 1_\pi \succ u_4 q = -\lambda u_4 q,\ u_1 1_\pi \bullet u_4 q = \lambda u_4 q,$$
$$u_2 1_\pi \prec u_1 q = -\lambda u_2 q,\ u_2 1_\pi \succ u_1 q = \lambda u_2 q + p_1 u_3 q + \frac{p_1 p_2}{\lambda + p_2} u_4 q,$$
$$u_2 1_\pi \bullet u_1 q = \lambda u_2 q,\ u_2 1_\pi \prec u_2 q = -\lambda u_1 q,$$
$$u_2 1_\pi \succ u_2 q = \lambda u_1 q - p_3 u_4 q - \frac{p_1 p_2}{\lambda + p_2} u_3 q, u_2 1_\pi \bullet u_2 q = \lambda u_1 q,$$
$$u_2 1_\pi \prec u_3 q = -\lambda u_4 q,\ u_2 1_\pi \succ u_3 q = \lambda u_4 q,$$

Examples

$$u_2 1_\pi \bullet u_3 q = \lambda u_4 q,\ u_2 1_\pi \prec u_4 q = -\lambda u_3 q,$$
$$u_2 1_\pi \succ u_4 q = \lambda u_3 q,\ u_2 1_\pi \bullet u_4 q = \lambda u_3 q,$$
$$u_3 1_\pi \prec u_1 q = -\lambda u_3 q,\ u_3 1_\pi \succ u_1 q = -(\lambda + p_2) u_3 q - p_2 u_4 q,$$
$$u_3 1_\pi \bullet u_1 q = \lambda u_3 q,\ u_3 1_\pi \prec u_2 q = \lambda u_4 q,$$
$$u_3 1_\pi \succ u_2 q = (\lambda + p_2) u_4 q + p_2 u_3 q,\ u_3 1_\pi \bullet u_2 q = -\lambda u_4 q,$$
$$u_4 1_\pi \prec u_1 q = -\lambda u_4 q,\ u_4 1_\pi \succ_{1_\pi, q} u_1 q = (\lambda + p_2) u_3 q + p_2 u_4 q,$$
$$u_4 1_\pi \bullet u_1 q = \lambda u_4 q,\ u_4 1_\pi \prec u_2 q = \lambda u_3 q,$$
$$u_4 1_\pi \succ u_2 q = -(\lambda + p_2) u_4 q - p_2 u_3 q,\ u_4 1_\pi \bullet u_2 q = -\lambda u_3 q,$$
$$u_1 q \prec u_1 1_\pi = -\lambda u_1 q,\ u_1 q \succ u_1 1_\pi = -\lambda u_1 q,$$
$$u_1 q \bullet u_1 1_\pi = \lambda u_1 q,\ u_1 q \prec u_2 1_\pi = \lambda u_2 q + p_1 u_3 q + \frac{p_1 p_2}{\lambda + p_2} u_4 q,$$
$$u_1 q \succ u_2 1_\pi = -\lambda u_2 q,\ u_1 q \bullet u_2 1_\pi = \lambda u_2 q,$$

Examples

$$u_1q \prec u_3 1_\pi = -(\lambda + p_2)u_3q - p_2u_4q,\ u_1q \succ u_3 1_\pi = -\lambda u_3q,$$
$$u_1q \bullet u_3 1_\pi = \lambda u_3q,\ u_1q \prec u_4 1_\pi = (\lambda + p_2)u_3q + p_2u_4q,$$
$$u_1q \succ u_4 1_\pi = -\lambda u_4q,\ u_1q \bullet u_4 1_\pi = \lambda u_4q,$$
$$u_2q \prec u_1 1_\pi = -\lambda u_2q,\ u_2q \succ u_1 1_\pi = -\lambda u_2q,$$
$$u_2q \bullet u_1 1_\pi = \lambda u_2q,\ u_2q \prec u_2 1_\pi = \lambda u_1q + p_1u_4q + \frac{p_1p_2}{\lambda + p_2}u_3q,$$
$$u_2q \succ u_2 1_\pi = -\lambda u_1q,\ u_2q \bullet u_2 1_\pi = \lambda u_1q,$$
$$u_2q \prec u_3 1_\pi = -(\lambda + p_2)u_4q - p_2u_3q,\ u_2q \succ u_3 1_\pi = -\lambda u_4q,$$
$$u_2q \bullet u_3 1_\pi = \lambda u_4q,\ u_2q \prec u_4 1_\pi = (\lambda + p_2)u_4q + p_2u_3q,$$
$$u_2q \succ u_4 1_\pi = -\lambda u_3q,\ u_2q \bullet u_4 1_\pi = \lambda u_3q,$$
$$u_3q \prec u_1 1_\pi = -\lambda u_3q,\ u_3q \succ u_1 1_\pi = -\lambda u_3q,$$
$$u_3q \bullet u_1 1_\pi = \lambda u_3q,\ u_3q \prec u_2 1_\pi = -\lambda u_4q,$$

Examples

$$u_3q \succ u_2 1_\pi = \lambda u_4 q,\ u_3 q \bullet u_2 1_\pi = -\lambda u_4 q,$$
$$u_4 q \prec u_1 1_\pi = -\lambda u_4 q,\ u_4 q \succ u_1 1_\pi = -\lambda u_4 q,$$
$$u_4 q \bullet u_1 1_\pi = \lambda u_4 q,\ u_4 q \prec u_2 1_\pi = -\lambda u_3 q,$$
$$u_4 q \succ u_2 1_\pi = \lambda u_3 q,\ u_4 q \bullet u_2 1_\pi = -\lambda u_3 q,$$
$$u_1 1_\pi \prec u_1 1_\pi = -\lambda u_1 1_\pi,\ u_1 1_\pi \succ u_1 1_\pi = -\lambda u_1 1_\pi,$$
$$u_1 1_\pi \bullet u_1 1_\pi = \lambda u_1 1_\pi,\ u_1 1_\pi \prec u_2 1_\pi = \lambda u_2 1_\pi + p_1 u_3 1_\pi + \frac{p_1 p_2}{\lambda + p_2} u_4 1_\pi,$$
$$u_1 1_\pi \succ u_2 1_\pi = -\lambda u_2 1_\pi,\ u_1 1_\pi \bullet u_2 1_\pi = \lambda u_2 1_\pi,$$
$$u_1 1_\pi \prec u_3 1_\pi = -(\lambda + p_2) u_3 1_\pi - p_2 u_4 1_\pi,\ u_1 1_\pi \succ u_3 1_\pi = -\lambda u_3 1_\pi,$$
$$u_1 1_\pi \bullet u_3 1_\pi = \lambda u_3 1_\pi,\ u_1 1_\pi \prec u_4 1_\pi = (\lambda + p_2) u_3 1_\pi + p_2 u_4 1_\pi,$$
$$u_1 1_\pi \succ u_4 1_\pi = -\lambda u_4 1_\pi,\ u_1 1_\pi \bullet u_4 1_\pi = \lambda u_4 1_\pi,$$

Examples

$$u_2 1_\pi \prec u_1 1_\pi = -\lambda u_2 1_\pi,\ u_2 1_\pi \succ u_1 1_\pi = \lambda u_2 1_\pi + p_1 u_3 1_\pi + \frac{p_1 p_2}{\lambda + p_2} u_4 1_\pi,$$

$$u_2 1_\pi \bullet u_1 1_\pi = \lambda u_2 1_\pi,\ u_2 1_\pi \prec u_2 1_\pi = \lambda u_1 1_\pi + p_1 u_4 1_\pi + \frac{p_1 p_2}{\lambda + p_2} u_3 1_\pi,$$

$$u_2 1_\pi \succ u_2 1_\pi = \lambda u_1 1_\pi - p_1 u_4 1_\pi - \frac{p_1 p_2}{\lambda + p_2} u_3 1_\pi, u_2 1_\pi \bullet u_2 1_\pi = \lambda u_1 1_\pi,$$

$$u_2 1_\pi \prec u_3 1_\pi = -(\lambda + p_2) u_4 1_\pi + p_2 u_3 1_\pi,\ u_2 1_\pi \succ u_3 1_\pi = \lambda u_4 1_\pi,$$

$$u_2 1_\pi \bullet u_3 1_\pi = \lambda u_4 1_\pi,\ u_2 1_\pi \prec u_4 1_\pi = (\lambda + p_2) u_4 1_\pi + p_2 u_3 1_\pi,$$

$$u_2 1_\pi \succ u_4 1_\pi = \lambda u_3 1_\pi,\ u_2 1_\pi \bullet u_4 1_\pi = \lambda u_3 1_\pi,$$

$$u_3 1_\pi \prec u_1 1_\pi = -\lambda u_3 1_\pi,\ u_3 1_\pi \succ u_1 1_\pi = -(\lambda + p_2) u_3 1_\pi - p_2 u_4 1_\pi,$$

$$u_3 1_\pi \bullet u_1 1_\pi = \lambda u_3 1_\pi,\ u_3 1_\pi \prec u_2 1_\pi = -\lambda u_4 1_\pi,$$

$$u_3 1_\pi \succ u_2 1_\pi = (\lambda + p_2) u_4 1_\pi + p_2 u_3 1_\pi, u_3 1_\pi \bullet_{1_\pi, 1_\pi} u_2 1_\pi = -\lambda u_4 1_\pi,$$

$$u_4 1_\pi \prec u_1 1_\pi = -\lambda u_4 1_\pi,\ u_4 1_\pi \succ u_1 1_\pi = (\lambda + p_2) u_3 1_\pi + p_2 u_4 1_\pi,$$

Examples

$$u_4 1_\pi \bullet u_1 1_\pi = \lambda u_4 1_\pi,\ u_4 1_\pi \prec u_2 1_\pi = -\lambda u_3 1_\pi,$$
$$u_4 1_\pi \succ u_2 1_\pi = -(\lambda + p_2) u_4 1_\pi - p_2 u_3 1_\pi, u_4 1_\pi \bullet u_2 1_\pi = -\lambda u_3 1_\pi,$$
$$u_1 q \prec u_1 q = -\lambda u_1 q, u_1 q \succ u_1 q = -\lambda u_1 q,$$
$$u_1 q \bullet u_1 q = \lambda u_1 q,\ u_1 q \prec u_2 q = -\lambda u_2 q,$$
$$u_1 q \succ u_2 q = -\lambda u_2 q,\ u_1 q \bullet u_2 q = \lambda u_2 q,$$
$$u_1 q \prec u_3 q = -\lambda u_3 q, u_1 q \succ u_3 q = -\lambda u_3 q,$$
$$u_1 q \bullet u_3 q = \lambda u_3 q,\ u_1 q \prec u_4 q = -\lambda u_4 q,$$
$$u_1 q \succ u_4 q = -\lambda u_4 q,\ u_1 q \bullet u_4 q = \lambda u_4 q,$$
$$u_2 q \prec u_1 q = -\lambda u_2 q, u_2 q \succ u_1 q = -\lambda u_2 q,$$
$$u_2 q \bullet u_1 q = \lambda u_2 q,\ u_2 q \prec u_2 q = -\lambda u_1 q,$$
$$u_2 q \succ u_2 q = -\lambda u_1 q,\ u_2 q \bullet u_2 q = \lambda u_1 q,$$

Examples

$$u_2q \prec u_3q = -\lambda u_4q, u_2q \succ u_3q = -\lambda u_4q,$$
$$u_2q \bullet u_3q = \lambda u_4q,\ u_2q \prec u_4q = -\lambda u_3q,$$
$$u_2q \succ u_4q = -\lambda u_3q,\ u_2q \bullet u_4q = \lambda u_3q,$$
$$u_3q \prec u_1q = -\lambda u_3q,\ u_3q \succ u_1q = -\lambda u_3q,$$
$$u_3q \bullet u_1q = \lambda u_3q, u_3q \prec u_2q = \lambda u_4q,$$
$$u_3q \succ u_2q = \lambda u_4q,\ u_3q \bullet u_2q = -\lambda u_4q$$
$$u_4q \prec u_1q = -\lambda u_4q,\ u_4q \succ u_1q = -\lambda u_4q,$$
$$u_4q \bullet u_1q = \lambda u_4q, u_4q \prec u_2q = \lambda u_3q,$$
$$u_4q \succ u_2q = \lambda u_3q,\ u_4q \bullet u_2q = -\lambda u_3q.$$

The operations for the remaining cases are 0.

Examples

(3) Based on Proposition 2.24 and (1), (2) above, we can get new structures of dendriform T-algebra on $\mathcal{A}[\pi]$, where $\mathcal{A}$ are given in Example 2.12 and Example 2.14, respectively.

Corollaries

Corollary 2.27

A Rota-Baxter algebra $(\mathcal{A}, \mathcal{R})$ *induces a dendriform algebra* $(\mathcal{A}, \prec, \succ)$, *where*

$$a \prec b := a\mathcal{R}(b) + \lambda ab, \quad a \succ b := \mathcal{R}(a)b$$

for $a, b \in A$.

Remark 2.28

If $\lambda = 0$ in Corollary 2.27, then we obtain [1, Proposition 4.5].

From (tri)dendriform T-algebras to T-algebras

Proposition 2.29

(1) Let $(A, \prec, \succ)$ *be a dendriform T-algebra. Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\diamond_{p,q}\}_{p,q\in\pi})$ *is a T-algebra, where* $\{\diamond_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$,

$$a \diamond_{p,q} b := a \prec_{p,q} b + a \succ_{p,q} b, \quad \textit{for } a \in A_p, b \in A_q.$$

(2) Let $(A, \prec, \succ, \bullet)$ *be a tridendriform T-algebra. Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\diamond_{p,q}\}_{p,q\in\pi})$ *is a T-algebra, where* $\{\diamond_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$,

$$a \diamond_{p,q} b := a \prec_{p,q} b + a \bullet_{p,q} b + a \succ_{p,q} b, \quad \textit{for } a \in A_p, b \in A_q.$$

Remark

Remark 2.30

We can get many new T-algebra structures on $\mathcal{A}[\pi]$ by Proposition 2.29 and Example 2.26.

Examples

Example 2.31

(1) In general, according to Part (1) in Proposition 2.29, commutative dendriform T-algebras induce commutative T-algebras. The following example comes from (3) in Example 2.26 which shows that noncommutative dendriform T-algebra can also induce commutative T-algebra.

The new structure of T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.12) can be defined by

$$u_1 1_\pi \diamond u_1 q = u_1 q \diamond u_1 1_\pi = -\lambda u_2 q, u_1 1_\pi \diamond u_2 q = u_2 q \diamond u_1 1_\pi = -\lambda u_2 q,$$
$$u_2 1_\pi \diamond u_1 q = u_1 q \diamond u_2 1_\pi = -\lambda u_2 q, u_2 1_\pi \diamond u_2 q = u_2 q \diamond u_2 1_\pi = -\lambda u_2 q,$$
$$u_1 1_\pi \diamond u_1 1_\pi = -\lambda u_1 1_\pi, u_1 1_\pi \diamond u_2 1_\pi = u_2 1_\pi \diamond u_1 1_\pi = -\lambda u_2 1_\pi,$$
$$u_2 1_\pi \diamond u_2 1_\pi = -\lambda u_2 1_\pi, u_1 q \diamond u_1 q = \lambda u_1 q - 2\lambda u_2 q,$$
$$u_2 q \diamond u_2 q = -\lambda u_2 q, u_1 q \diamond u_2 q = u_2 q \diamond u_1 q = -\lambda u_2 q.$$

Examples

(2) The following example comes from (4) in Example 2.26 which shows that noncommutative dendriform algebra can also induce noncommutative T-algebra.
The new structure of T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.14) can be defined by

$$u_1 1_\pi \diamond u_1 q = u_1 q \diamond u_1 1_\pi = -\lambda u_1 q, u_1 1_\pi \diamond u_2 q = u_2 q \diamond u_1 1_\pi = -\lambda u_2 q,$$
$$u_1 1_\pi \diamond u_3 q = u_3 q \diamond u_1 1_\pi = -\lambda u_3 q, u_1 1_\pi \diamond u_4 q = u_4 q \diamond u_1 1_\pi = -\lambda u_4 q,$$
$$u_2 1_\pi \diamond u_1 q = u_1 q \diamond u_2 1_\pi = \lambda u_2 q + p_1 u_3 q + \frac{p_1 p_2}{\lambda + p_2} u_4 q, u_2 1_\pi \diamond u_4 q = \lambda u_3 q,$$
$$u_2 1_\pi \diamond u_2 q = \lambda u_1 q - p_3 u_4 q - \frac{p_1 p_2}{\lambda + p_2} u_3 q, u_2 1_\pi \diamond u_3 q = \lambda u_4 q,$$
$$u_4 q \diamond u_2 1_\pi = -\lambda u_3 q, u_3 1_\pi \diamond u_1 q = u_1 q \diamond u_3 1_\pi = -(\lambda + p_2) u_3 q - p_2 u_4 q,$$
$$u_2 q \diamond u_2 1_\pi = \lambda u_1 q + p_1 u_4 q + \frac{p_1 p_2}{\lambda + p_2} u_3 q, u_3 q \diamond u_2 1_\pi = -\lambda u_4 q,$$

Examples

$$u_3 1_\pi \diamond u_2 q = -u_2 q \diamond u_3 1_\pi = (\lambda + p_2) u_4 q + p_2 u_3 q, u_1 1_\pi \diamond u_1 1_\pi = -\lambda u_1 1_\pi,$$
$$u_4 1_\pi \diamond u_1 q = u_1 q \diamond u_4 1_\pi = (\lambda + p_2) u_3 q + p_2 u_4 q,$$
$$-u_4 1_\pi \diamond u_2 q = u_2 q \diamond u_4 1_\pi = (\lambda + p_2) u_4 q + p_2 u_3 q,$$
$$u_2 1_\pi \diamond u_3 1_\pi = (\lambda - p_2) u_4 1_\pi + p_2 u_3 1_\pi,$$
$$u_1 1_\pi \diamond u_2 1_\pi = u_2 1_\pi \diamond u_1 1_\pi = \lambda u_2 1_\pi + p_1 u_3 1_\pi + \frac{p_1 p_2}{\lambda + p_2} u_4 1_\pi,$$
$$u_1 1_\pi \diamond u_3 1_\pi = u_3 1_\pi \diamond u_1 1_\pi = -(\lambda + p_2) u_3 1_\pi - p_2 u_4 1_\pi,$$
$$u_2 1_\pi \diamond u_4 1_\pi = -u_4 1_\pi \diamond u_2 1_\pi = (\lambda + p_2) u_4 1_\pi + (2\lambda + p_2) u_3 1_\pi,$$
$$u_4 1_\pi \diamond u_1 1_\pi = u_1 1_\pi \diamond u_4 1_\pi = (\lambda + p_2) u_3 1_\pi + p_2 u_4 1_\pi,$$
$$u_1 q \diamond u_2 q = u_2 q \diamond u_1 q = -\lambda u_2 q, u_1 q \diamond u_3 q = u_3 q \diamond u_1 q = -\lambda u_3 q,$$
$$u_1 q \diamond u_4 q = u_4 q \diamond u_1 q = -\lambda u_4 q, \ -u_2 q \diamond u_4 q = u_4 q \diamond u_2 q = \lambda u_3 q,$$
$$u_1 q \diamond u_1 q = -\lambda u_1 q, u_2 q \diamond u_2 q = -\lambda u_1 q, u_2 1_\pi \diamond u_2 1_\pi = 3\lambda u_1 1_\pi,$$
$$-u_2 q \diamond u_3 q = u_3 q \diamond u_2 q = \lambda u_4 q, u_3 1_\pi \diamond u_2 1_\pi = (-\lambda + p_2) u_4 1_\pi + p_2 u_3 1_\pi.$$

The operations for the remaining cases are 0.

From Rota-Baxter T-algebras to T-algebras

Corollary 2.32

Let (A, R) be a Rota-Baxter T-algebra. If we define a new multiplication $\{\diamond_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ by

$$a \diamond_{p,q} b = a \cdot_{p,q} R(b) + R(a) \cdot_{p,q} b + \lambda a \cdot_{p,q} b$$

for $a \in A_p$, $b \in A_q$ and $p, q \in \pi$. Then $(\{A_\varphi\}_{\varphi\in\pi}, \{\diamond_{p,q}\}_{p,q\in\pi})$ is a T-algebra.

Remark 2.33

For a Rota-Baxter T-algebra (A, R), we have $R_p(a) \cdot_{p,q} R_q(b) = R_{pq}(a \diamond_{p,q} b)$.

Rota-Baxter Hopf T-algebras

Definition 3.1

Let π be a semigroup. A Rota-Baxter T-algebra (A, R) is a **Rota-Baxter semi-Hopf T-algebra** denoted by (A, Δ, R) if every $\{A_\varphi\}_{\varphi\in\pi}$ is a coalgebra with comultiplication Δ_φ such that $\{\mu_{p,q}\}_{p,q\in\pi}$ are coalgebra maps and R_φ is a Rota-Baxter cooperator.

If, furthermore, π is a monoid with unit 1, and Rota-Baxter T-algebra is unital such that $\eta : K \longrightarrow A_1$ is a coalgebra map, then we call Rota-Baxter semi-Hopf T-algebra **unital**.

Let π be a group. A **Rota-Baxter Hopf T-algebra** denoted by (A, Δ, R, S) is a unital Rota-Baxter semi-Hopf T-algebra together with a family of linear maps $\{S_\varphi : A_\varphi \longrightarrow A_{\varphi^{-1}}\}_{\varphi\in\pi}$ such that

$$\mu_{\varphi^{-1},\varphi} \circ (S_\varphi \otimes id_{A_\varphi}) \circ \Delta_\varphi = \eta\varepsilon_\varphi = \mu_{\varphi,\varphi^{-1}} \circ (id_{A_\varphi} \otimes S_\varphi) \circ \Delta_\varphi,$$
$$S_\varphi \circ R_\varphi = R_{\varphi^{-1}} \circ S_\varphi.$$

Rota-Baxter Hopf T-algebras

Remark 3.2

(1) Rota-Baxter Hopf T-algebra (A, Δ, R, S) includes a Hopf T-algebra (A, Δ, S).
(2) If $(\mathcal{A}, \widetilde{\mu}, \widetilde{\eta}, \widetilde{\Delta}, \widetilde{\varepsilon}, \mathcal{S})$ is a Hopf algebra, then we call Rota-Baxter Hopf T-algebra $(\{A_\varphi = \mathcal{A}\}, \{\mu_{p,q} = \widetilde{\mu}\}, \{R_\varphi\}, \lambda, \{\Delta_\varphi = \widetilde{\Delta}\}, \{\varepsilon_\varphi = \widetilde{\varepsilon}\}, \{S_\varphi = \mathcal{S}\})$ a **Rota-Baxter family Hopf algebra of weight** λ.
(3) If $\pi = \{1\}$, then we call Rota-Baxter Hopf T-algebra $(A_1, \mu_{1,1}, R_1, \lambda, \Delta_1, \varepsilon_1, S_1)$ is a (R_1, R_1)**-Rota-Baxter Hopf algebra of weight** λ. We note here (R_1, R_1)-Rota-Baxter Hopf algebra of weight λ is a (R_1, R_1)-Rota-Baxter bialgebra of weight λ together with antipode S_1 such that $S_1 \circ R_1 = R_1 \circ S_1$.

Examples

Example 3.3

Let $\pi = \{1, q\}$ be a monoid with a unit 1 and $q^2 = q$.
(1) Let $\mathcal{A}$ be the 2-dimensional algebra defined in Example 2.12. For all $\varphi \in \pi$, define $\Delta_\varphi : \mathcal{A}\varphi \longrightarrow \mathcal{A}\varphi$ by $\Delta(h\varphi) = h\varphi \otimes h\varphi$, $\forall\, h \in \mathcal{A}$. Then $(\mathcal{A}[\pi], \Delta, R)$ is a Rota-Baxter semi-Hopf T-algebra of weight λ with $\{R_\varphi\}$ given by

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto 0q, \\ u_2 q &\longmapsto -\lambda u_2 q. \end{aligned}$$

Examples

(2) Let $\mathcal{A}$ be the 3-dimensional algebra defined in Example 2.13. For all $\varphi \in \pi$, define $\Delta_\varphi : \mathcal{A}\varphi \longrightarrow \mathcal{A}\varphi$ by $\Delta(h\varphi) = h\varphi \otimes h\varphi$, $\forall\, h \in \mathcal{A}$. Then $(\mathcal{A}[\pi], \Delta, R)$ is a Rota-Baxter semi-Hopf T-algebra of weight λ with $\{R_\varphi\}$ given by

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto 0q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto 0q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto 0q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto 0 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

Examples

or

$$
\begin{aligned}
R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\
u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\
u_2 1_\pi &\longmapsto 0 1_\pi \\
u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi
\end{aligned}
\qquad
\begin{aligned}
R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\
u_1 q &\longmapsto -\lambda u_1 q, \\
u_2 q &\longmapsto -\lambda u_2 q, \\
u_3 q &\longmapsto -\lambda u_3 q
\end{aligned}
$$

or

$$
\begin{aligned}
R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\
u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\
u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\
u_3 1_\pi &\longmapsto 0 1_\pi
\end{aligned}
\qquad
\begin{aligned}
R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\
u_1 q &\longmapsto 0 q, \\
u_2 q &\longmapsto -\lambda u_2 q, \\
u_3 q &\longmapsto 0 q
\end{aligned}
$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto 0q, \\ u_2 q &\longmapsto 0q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto 0q, \\ u_2 q &\longmapsto 0q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto 0q, \\ u_2 q &\longmapsto 0q, \\ u_3 q &\longmapsto -\lambda u_3 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto 0q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q. \end{aligned}$$

Examples

(3) Let $\mathcal{A}$ be the 4-dimensional algebra defined in Example 2.14. For all $\varphi \in \pi$, define $\Delta_\varphi : \mathcal{A}\varphi \longrightarrow \mathcal{A}\varphi$ by $\Delta(h\varphi) = h\varphi \otimes h\varphi$, $\forall\, h \in \mathcal{A}$. Then $(\mathcal{A}[\pi], \Delta, R)$ is a Rota-Baxter semi-Hopf T-algebra of weight λ with $\{R_\varphi\}$ given by

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi \\ u_1 1_\pi &\longmapsto 0 1_\pi \\ u_2 1_\pi &\longmapsto 0 1_\pi \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi \\ u_4 1_\pi &\longmapsto -\lambda u_4 1_\pi \end{aligned} \qquad \begin{aligned} R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 q &\longmapsto -\lambda u_3 q, \\ u_4 q &\longmapsto -\lambda u_4 q \end{aligned}$$

Examples

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi & R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi & u_1 q &\longmapsto -\lambda u_1 q, \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi & u_2 q &\longmapsto -\lambda u_2 q, \\ u_3 1_\pi &\longmapsto 0 1_\pi & u_3 q &\longmapsto -\lambda u_3 q, \\ u_4 1_\pi &\longmapsto 0 1_\pi & u_4 q &\longmapsto -\lambda u_4 q \end{aligned}$$

or

$$\begin{aligned} R_{1_\pi} : \mathcal{A}1_\pi &\longrightarrow \mathcal{A}1_\pi & R_q : \mathcal{A}q &\longrightarrow \mathcal{A}q \\ u_1 1_\pi &\longmapsto -\lambda u_1 1_\pi & u_1 q &\longmapsto 0q, \\ u_2 1_\pi &\longmapsto -\lambda u_2 1_\pi & u_2 q &\longmapsto 0q, \\ u_3 1_\pi &\longmapsto -\lambda u_3 1_\pi & u_3 q &\longmapsto -\lambda u_3 q, \\ u_4 1_\pi &\longmapsto -\lambda u_4 1_\pi & u_4 q &\longmapsto -\lambda u_4 q. \end{aligned}$$

Rota-Baxter Hopf T-coalgebras

Definition 3.4

Let π be a semigroup and $\gamma \in K$ be given. A **Rota-Baxter T-coalgebra of weight** γ is a quadruple $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{p,q}\}_{p,q\in\pi}, \{Q_\varphi\}_{\varphi\in\pi}, \gamma)$ $(abbr.(C, Q))$, where $\{C_\varphi\}_{\varphi\in\pi}$ is a family of vector spaces together with a family of linear maps $\{\Delta_{p,q} : C_{pq} \longrightarrow C_p \otimes C_q\}_{p,q\in\pi}$ and a family of linear maps $\{Q_\varphi : C_\varphi \longrightarrow C_\varphi\}_{\varphi\in\pi}$ such that

$$(\Delta_{p,q} \otimes id_{C_t})\Delta_{pq,t} = (id_{C_p} \otimes \Delta_{q,t})\Delta_{p,qt}, \tag{12}$$

$$(Q_p \otimes Q_q)\Delta_{p,q} = (id_{C_p} \otimes Q_q)\Delta_{p,q}Q_{p,q} + (Q_p \otimes id_{C_q})\Delta_{p,q}Q_{p,q} + \gamma\Delta_{p,q}Q_{p,q},$$

where $p, q, t \in \pi$.

Rota-Baxter Hopf T-coalgebras

If moreover π is a monoid with unit 1 and there is a linear map $\varepsilon : C_1 \longrightarrow K$ such that

$$(id_{C_p} \otimes \varepsilon)\Delta_{p,1} = id_{C_p} = (\varepsilon \otimes id_{C_p})\Delta_{1,p},$$

then we call (C, Q) **counital.**

Rota-Baxter Hopf T-coalgebras

Remark 3.5

(1) If the semigroup π contains a single element 1, then Rota-Baxter T-coalgebra $(C_1, \Delta_{1,1}, Q_1, \gamma)$ is exactly the Rota-Baxter coalgebra of weight λ.
(2) If (C, Δ, ε) is a coassociative coalgebra, then Rota-Baxter T-coalgebra $(\{C_\varphi = C\}, \{\Delta_{p,q} = \Delta\}, \{Q_\varphi\}, \gamma)$ is called a **Rota-Baxter family coalgebra of weight** γ, which is a dual to the Rota-Baxter family algebra of weight λ.
(3) Eq.(12) means that Rota-Baxter T-coalgebra (C, Q) includes a T-coalgebra $(\{C_\varphi\}, \{\Delta_{p,q}\})$.

Dendriform T-coalgebras

Definition 3.6

Let π be a semigroup. A **dendriform T-coalgebra** is a family of vector spaces $\{C_\varphi\}_{\varphi\in\pi}$ with a family of binary operations $\{\Delta_{\prec p,q}, \Delta_{\succ p,q} : C_{pq} \longrightarrow C_p \otimes C_q\}_{p,q\in\pi}$ satisfying the following conditions (for all $p, q, t \in \pi$)

$$(\Delta_{\prec p,q} \otimes id_{C_t})\Delta_{\prec pq,t} = (id_{C_p} \otimes \Delta_{\prec q,t})\Delta_{\prec p,qt} + (id_{C_p} \otimes \Delta_{\succ q,t})\Delta_{\prec p,qt},$$

$$(\Delta_{\succ p,q} \otimes id_{C_t})\Delta_{\prec pq,t} = (id_{C_p} \otimes \Delta_{\prec q,t})\Delta_{\succ p,qt},$$

$$(\Delta_{\prec p,q} \otimes id_{C_t})\Delta_{\succ pq,t} + (\Delta_{\succ p,q} \otimes id_{C_t})\Delta_{\succ pq,t} = (id_{C_p} \otimes \Delta_{\succ q,t})\Delta_{\succ p,qt}.$$

For simplicity, we denote it by $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi})$ (abbr. $(C, \Delta_\prec, \Delta_\succ)$).

Dendriform T-coalgebras

Remark 3.7

If π contains a single element 1, then $(C_1, \Delta_{\prec p,q} = \Delta_{\prec 1,1}, \Delta_{\succ p,q} = \Delta_{\succ 1,1})$ is dendriform coalgebra.

Tridendriform T-coalgebra

Definition 3.8

Let π be a semigroup. A **tridendriform T-coalgebra** is a family of vector spaces $\{C_\varphi\}_{\varphi\in\pi}$ with a family of binary operations $\{\Delta_{\prec p,q}, \Delta_{\succ p,q}, \Delta_{\bullet p,q} : C_{pq} \longrightarrow C_p \otimes C_q\}_{p,q\in\pi}$ satisfying the following conditions (for all $p, q, t \in \pi$)

$$(\Delta_{\prec p,q} \otimes id_{C_t})\Delta_{\prec pq,t}$$

$$= (id_{C_p} \otimes \Delta_{\prec q,t})\Delta_{\prec p,qt} + (id_{C_p} \otimes \Delta_{\succ q,t})\Delta_{\prec p,qt} + (id_{C_p} \otimes \Delta_{\bullet q,t})\Delta_{\prec p,qt},$$

$$(\Delta_{\succ p,q} \otimes id_{C_t})\Delta_{\prec pq,t} = (id_{C_p} \otimes \Delta_{\prec q,t})\Delta_{\succ p,qt},$$

$$(\Delta_{\prec p,q} \otimes id_{C_t})\Delta_{\succ pq,t} + (\Delta_{\succ p,q} \otimes id_{C_t})\Delta_{\succ pq,t} + (\Delta_{\bullet p,q} \otimes id_{C_t})\Delta_{\succ pq,t}$$

$$= (id_{C_p} \otimes \Delta_{\succ q,t})\Delta_{\succ p,qt},$$

$$(\Delta_{\succ p,q} \otimes id_{C_t})\Delta_{\bullet pq,t} = (id_{C_p} \otimes \Delta_{\bullet q,t})\Delta_{\succ p,qt},$$

$$(\Delta_{\prec p,q} \otimes id_{C_t})\Delta_{\bullet pq,t} = (id_{C_p} \otimes \Delta_{\succ q,t})\Delta_{\bullet p,qt},$$

$$(\Delta_{\bullet p,q} \otimes id_{C_t})\Delta_{\prec pq,t} = (id_{C_p} \otimes \Delta_{\prec q,t})\Delta_{\bullet p,qt},$$

$$(\Delta_{\bullet p,q} \otimes id_{C_t})\Delta_{\bullet pq,t} = (id_{C_p} \otimes \Delta_{\bullet q,t})\Delta_{\bullet p,qt}.$$

Tridendriform T-coalgebra

For simplicity, we denote it by $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi}, \{\Delta_{\bullet p,q}\}_{p,q\in\pi})$ (abbr. $(C, \Delta_\prec, \Delta_\succ, \Delta_\bullet)$).

Tridendriform T-coalgebra

Remark 3.9

If π contains a single element 1, then $(C_1, \Delta_{\prec p,q} = \Delta_{\prec 1,1}, \Delta_{\succ p,q} = \Delta_{\succ 1,1}, \Delta_{\bullet p,q} = \Delta_{\bullet 1,1})$ is tridendriform coalgebra.

Relations

Proposition 3.10

Let π be a semigroup. (1) A Rota-Baxter T-coalgebra (C, Q) induces a dendriform T-coalgebra $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi})$, where

$$\Delta_{\prec p,q}(c) = c_{(1,p)} \otimes Q_q(c_{(2,q)}) + \gamma c_{(1,p)} \otimes c_{(2,q)}, \ \Delta_{\succ p,q}(c) = Q_p(c_{(1,p)}) \otimes c_{(2,q)},$$

here we write $\Delta_{p,q}(c) = c_{(1,p)} \otimes c_{(2,q)}$, for all $c \in C_{pq}$ and $p, q \in \pi$.

Relations

(2) A Rota-Baxter T-coalgebra (C, Q) induces a tridendriform T-coalgebra $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi}, \{\Delta_{\bullet p,q}\}_{p,q\in\pi})$, where

$$\Delta_{\prec p,q}(c) = c_{(1,p)} \otimes Q_q(c_{(2,q)}),\ \Delta_{\succ p,q}(c) = Q_p(c_{(1,p)}) \otimes c_{(2,q)},\ \Delta_{\bullet p,q}(c) = \gamma c_{(1,p)} \otimes c_{(2,q)},$$

here we write $\Delta_{p,q}(c) = c_{(1,p)} \otimes c_{(2,q)}$, for all $c \in C_{pq}$ and $p, q \in \pi$.

Relations

Proposition 3.11

Let π *be a semigroup.* *(1) Let* $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi})$ *be a dendriform T-coalgebra. Then* $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\diamond p,q}\}_{p,q\in\pi})$ *is a T-coalgebra, where* $\{\Delta_{\diamond p,q} : C_{pq} \longrightarrow C_p \otimes C_q\}_{p,q\in\pi}$,

$$\Delta_{\diamond p,q}(c) = \Delta_{\prec p,q}(c) + \Delta_{\succ p,q}(c)$$

for all $c \in C_{pq}$ *and* $p, q \in \pi$.

Relations

(2) Let $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\prec p,q}\}_{p,q\in\pi}, \{\Delta_{\succ p,q}\}_{p,q\in\pi}, \{\Delta_{\bullet p,q}\}_{p,q\in\pi})$ be a tridendriform T-coalgebra. Then $(\{C_\varphi\}_{\varphi\in\pi}, \{\Delta_{\diamond p,q}\}_{p,q\in\pi})$ is a T-coalgebra, where $\{\Delta_{\diamond p,q} : C_{pq} \longrightarrow C_p \otimes C_q\}_{p,q\in\pi}$,

$$\Delta_{\diamond p,q}(c) = \Delta_{\prec p,q}(c) + \Delta_{\bullet p,q}(c) + \Delta_{\succ p,q}(c)$$

for all $c \in C_{pq}$ and $p, q \in \pi$.

Rota-Baxter Hopf T-coalgebras

Definition 3.12

Let π be a semigroup. A Rota-Baxter T-coalgebra is a **Rota-Baxter semi-Hopf T-coalgebra** denoted by (C, μ, Q) if every $\{C_\varphi\}_{\varphi\in\pi}$ is an algebra with multiplication μ_φ such that $\{\Delta_{p,q}\}_{p,q\in\pi}$ are algebra maps and each Q_φ is a Rota-Baxter operator, i.e.,

$$Q_\varphi(a)Q_\varphi(b) = Q_\varphi(Q_\varphi(a)b + aQ_\varphi(b) + \gamma ab), \ \forall a, b \in C_\varphi.$$

Rota-Baxter Hopf T-coalgebras

If, moreover, π is a monoid with unit 1, and Rota-Baxter T-coalgebra is counital such that $\varepsilon : C_1 \longrightarrow K$ is an algebra map, then we call Rota-Baxter semi-Hopf T-coalgebra **counital.**
Let π be a group. A **Rota-Baxter Hopf T-coalgebra** denoted by (C, μ, Q, S) is a counital Rota-Baxter semi-Hopf T-coalgebra together with a family of linear maps $\{S_\varphi : C_\varphi \longrightarrow C_{\varphi^{-1}}\}_{\varphi\in\pi}$ such that

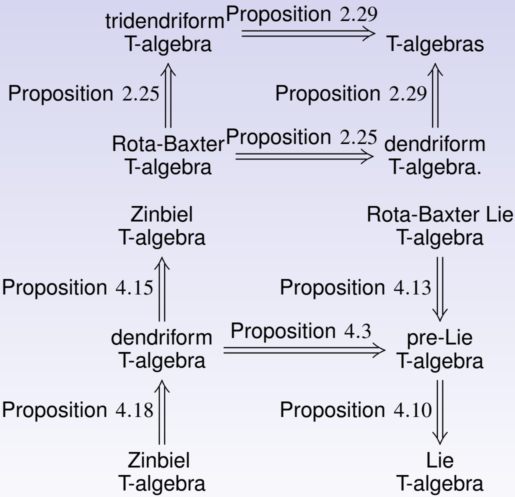
$$\mu_\varphi \circ (S_\varphi \otimes id_{C_\varphi}) \circ \Delta_{\varphi^{-1},\varphi} = \eta_\varphi \varepsilon = \mu_\varphi \circ (id_{C_\varphi} \otimes S_\varphi) \circ \Delta_{\varphi,\varphi^{-1}},$$
$$Q_{\varphi^{-1}} \circ S_\varphi = S_\varphi \circ Q_\varphi.$$

Rota-Baxter Hopf T-coalgebras

Remark 3.13

(1) Rota-Baxter Hopf T-coalgebra (C, μ, Q, S) includes a Hopf T-coalgebra $(\{C_\varphi\}, \{\Delta_{p,q}\}, \varepsilon, \mu_\varphi, \{\eta_\varphi\}, \varepsilon_\varphi, S_\varphi)$.
(2) If $(A, \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra, then we call Rota-Baxter Hopf T-coalgebra $(\{A_\varphi = A\}, \{\Delta_{p,q} = \Delta\}, \{Q_\varphi\}, \varepsilon, \gamma, \{\mu_\varphi = \mu\}, \{\eta_\varphi = \eta\}, \{S_\varphi = S\})$ a **co-Rota-Baxter family Hopf algebra of weight** γ.
(3) If $\pi = \{1\}$, then the Rota-Baxter Hopf T-coalgebra $(\{C_1\}, \{\Delta_1\}, \{Q_1\}, \varepsilon, \gamma, \{\mu_1\}, \{\eta_1\}, \{S_1\})$ is a (Q_1, Q_1)-Rota-Baxter Hopf algebra of weight γ.

T-versions



From now on we assume that the π is commutative semigroup.

Commutative T-algebras

Definition 4.1

A **commutative** T-algebra is a T-algebra $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi})$ satisfying

$$a \cdot_{p,q} b = b \cdot_{q,p} a, \tag{13}$$

for $a \in A_p$, $b \in A_q$, and $p, q \in \pi$.

Pre-Lie T-algebras

Definition 4.2

A **pre-Lie T-algebra** is a family of vector spaces $\{A_\varphi\}_{\varphi\in\pi}$ with a family of binary operations $\{*_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ such that

$$a *_{p,qt} (b *_{q,t} c) - (a *_{p,q} b) *_{pq,t} c = b *_{q,pt} (a *_{p,t} c) - (b *_{q,p} a) *_{qp,t} c,$$

for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$. We denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$.

From dendriform T-algebras to pre-Lie T-algebras

Proposition 4.3

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi})$ *be a dendriform T-algebra. Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *is a pre-Lie T-algebra, where* $\{*_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ *defined by*

$$a *_{p,q} b := a \succ_{p,q} b - b \prec_{q,p} a,$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$.

Remark

Remark 4.4

New constructions of pre-Lie T-algebra can be obtained by Proposition 4.3 and Example 2.26.

Examples

Example 4.5

(1) In general, according to Proposition 4.3, commutative dendriform T-algebras induce commutative pre-Lie T-algebras. The following example comes from (3) in Example 2.26 which shows that noncommutative dendriform T-algebra can also induce commutative pre-Lie T-algebra.
The new structure of pre-Lie T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.12) can be defined by

Examples

$$u_1 1_\pi * u_1 q = u_1 q * u_1 1_\pi = -\lambda u_1 q, \qquad u_1 1_\pi * u_2 q = u_2 q * u_1 1_\pi = -\lambda u_2 q,$$
$$u_2 1_\pi * u_1 q = u_1 q * u_2 1_\pi = -\lambda u_2 q, \qquad u_2 1_\pi * u_2 q = u_2 q * u_2 1_\pi = -\lambda u_2 q,$$
$$u_1 1_\pi * u_2 1_\pi = u_2 1_\pi * u_1 1_\pi = -\lambda u_2 1_\pi, \quad u_1 q * u_2 q = u_2 q * u_1 q = -\lambda u_2 q,$$
$$u_1 1_\pi * u_1 1_\pi = -\lambda u_1 1_\pi, \quad u_1 q * u_1 q = -\lambda u_1 q, \qquad u_2 1_\pi * u_2 1_\pi = -\lambda u_2 1_\pi,$$
$$u_2 q * u_2 q = -\lambda u_2 q.$$

Examples

(2) The following example comes from (4) in Example 2.26 which shows that noncommutative dendriform T-algebra can also induce noncommutative pre-Lie T-algebra.
The new structure of pre-Lie T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.14) can be defined by

$$u_1 1_\pi * u_1 q = u_1 q * u_1 1_\pi = -\lambda u_1 q, u_1 1_\pi * u_2 q = u_2 q * u_1 1_\pi = -\lambda u_2 q,$$

$$u_1 1_\pi * u_3 q = u_3 q * u_1 1_\pi = -\lambda u_3 q, u_1 1_\pi * u_4 q = u_4 q * u_1 1_\pi = -\lambda u_4 q,$$

$$u_2 1_\pi * u_1 q = u_1 q * u_2 1_\pi = -\lambda u_2 q,$$

$$u_2 1_\pi * u_2 q = -\lambda u_1 q - p_3 u_4 q - p_1 u_4 q - \frac{2p_1 p_2}{\lambda + p_2} u_3 q,$$

$$u_2 q * u_2 1_\pi = -\lambda u_1 q, u_2 1_\pi * u_3 q = 3\lambda u_4 q, u_3 q * u_2 1_\pi = \lambda u_4 q,$$

$$u_2 1_\pi * u_4 q = 3\lambda u_3 q, u_4 q * u_2 1_\pi = \lambda u_3 q, u_3 1_\pi * u_1 q = u_1 q * u_3 1_\pi = -\lambda u_3 q,$$

$$u_3 1_\pi * u_2 q = (\lambda + 2p_2) u_4 q + 2p_2 u_3 q, u_2 q * u_3 1_\pi = -\lambda u_4 q,$$

Examples

$$u_4 1_\pi * u_1 q = u_1 q * u_4 1_\pi = -\lambda u_4 q,$$
$$u_4 1_\pi * u_2 q = -2(\lambda + p_2) u_4 q - (\lambda + 2p_2) u_3 q,$$
$$u_2 q * u_4 1_\pi = -\lambda u_3 q, u_1 1_\pi * u_1 1_\pi = -\lambda u_1 1_\pi,$$
$$u_1 1_\pi * u_2 1_\pi = u_2 1_\pi * u_1 1_\pi = -\lambda u_2 1_\pi,$$
$$u_1 1_\pi * u_3 1_\pi = u_3 1_\pi * u_1 1_\pi = -\lambda u_3 1_\pi, u_1 1_\pi * u_4 1_\pi = u_4 1_\pi * u_1 1_\pi = -\lambda u_4 1_\pi,$$
$$u_2 1_\pi * u_2 1_\pi = -\lambda u_1 1_\pi - 2p_1 u_4 1_\pi - \frac{2p_1 p_2}{\lambda + p_2} u_3 1_\pi, u_2 1_\pi * u_3 1_\pi = 3\lambda u_4 1_\pi,$$
$$u_3 1_\pi * u_2 1_\pi = (\lambda + 2p_2) u_4 1_\pi, u_2 1_\pi * u_4 1_\pi = 3\lambda u_3 1_\pi,$$
$$u_4 1_\pi * u_2 1_\pi = -2(\lambda + p_2) u_4 1_\pi - (\lambda + 2p_2) u_3 1_\pi, u_1 q * u_1 q = -\lambda u_1 q,$$
$$u_1 q * u_2 q = u_2 q * u_1 q = -\lambda u_2 q, u_1 q * u_3 q = u_3 q * u_1 q = -\lambda u_3 q,$$
$$u_1 q * u_4 q = u_4 q * u_1 q = -\lambda u_4 q, u_2 q * u_2 q = -\lambda u_1 q,$$
$$u_2 q * u_3 q = -u_3 q * u_2 q = -\lambda u_4 q, u_2 q * u_4 q = -u_4 q * u_2 q = -\lambda u_3 q.$$

The operations for the remaining cases are 0.

Lie T-algebras

Definition 4.6

A **Lie T-algebra** is a family of vector spaces $\{A_\varphi\}_{\varphi\in\pi}$ together with a family of binary operations $\{[,]_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$, such that

$$[a,b]_{p,q} + [b,a]_{q,p} = 0,$$
$$[[a,b]_{p,q}, c]_{pq,t} + [[b,c]_{q,t}, a]_{qt,p} + [[c,a]_{t,p}, b]_{tp,q} = 0$$

for $a \in A_p$, $b \in A_q$, $c \in A_t$, and $p,q,t \in \pi$. We denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ (abbr. $(A,[,])$).

From T-algebras to Lie T-algebras

Proposition 4.7

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi})$ *be a T-algebra. Then* $(A, [,])$ *is a Lie T-algebra, where* $\{[,]_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$

$$[a, b]_{p,q} = a \cdot_{p,q} b - b \cdot_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$.

Remark 4.8

Based on Proposition 4.7 and Remark 2.30, we can get many new constructions of Lie T-algebra.

Examples

Example 4.9

According to Example 2.31 (2) and Proposition 4.7, by Eq.(14), the nonzero operations of Lie T-algebra on $\mathcal{A}[\pi]$ (where $\mathcal{A}$ is given in Example 2.14) can be defined by

$$
\begin{aligned}
&[u_2 1_\pi, u_2 q] = -p_3 u_4 q - p_1 u_4 q - \frac{2p_1 p_2}{\lambda + p_2} u_3 q,\\
&[u_2 1_\pi, u_3 q] = 2\lambda u_4 q, [u_2 1_\pi, u_4 q] = 2\lambda u_3 q,\\
&[u_3 1_\pi, u_2 q] = 2(\lambda + p_2) u_4 q + 2p_2 u_3 q,\\
&[u_4 1_\pi, u_2 q] = -2(\lambda + p_2) u_4 q - 2p_2 u_3 q,\\
&[u_2 1_\pi, u_3 1_\pi] = 2(\lambda - p_2) u_4 1_\pi,\\
&[u_2 1_\pi, u_4 1_\pi] = 2(\lambda + p_2) u_4 1_\pi + 2(2\lambda + p_2) u_3 1_\pi,\\
&[u_2 q, u_3 q] = -2\lambda u_4 q, \qquad [u_2 q, u_4 q] = -2\lambda u_3 q.
\end{aligned}
$$

From pre-Lie T-algebras to Lie T-algebras

Proposition 4.10

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *be a pre-Lie T-algebra. Define*

$$[a,b]_{p,q} = a *_{p,q} b - b *_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ *is a Lie T-algebra.*

Remark 4.11

According to Proposition 4.10 and Remark 4.4, we can get a series of construction of Lie T-algebra.

Rota-Baxter Lie T-algebras

Definition 4.12

Let $\lambda \in K$. A **Rota-Baxter Lie T-algebra of weight** λ is a Lie T-algebra $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ endowed with a family of linear maps $\{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$, subject to the relation

$$\begin{aligned}&[R_p(a), R_q(b)]_{p,q}\\ &= R_{pq}([a, R_q(b)]_{p,q}) + R_{pq}([R_p(a), b]_{p,q}) + \lambda R_{pq}([a,b]_{p,q})\end{aligned}$$

for $a \in A_p$, $b \in A_q$ and $p, q \in \pi$. We denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi}, \lambda)$.

From Rota-Baxter Lie T-algebras to pre-Lie T-algebras

Proposition 4.13

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi})$ *be a Rota-Baxter Lie T-algebra of weight* 0. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *is a pre-Lie T-algebra, where*

$$a *_{p,q} b = [R_p(a), b]_{p,q}.$$

Dendriform T-algebra and Zinbiel T-algebra

Definition 4.14

A **Zinbiel T-algebra** is a family of vector spaces $\{A_\varphi\}_{\varphi\in\pi}$ together with a family of binary operations $\{\star_{p,q} : A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q\in\pi}$ such that

$$a \star_{p,qt} (b \star_{q,t} c) = (a \star_{p,q} b) \star_{pq,t} c + (b \star_{q,p} a) \star_{qp,t} c$$

for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$. We denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$.

Dendriform T-algebra and Zinbiel T-algebra

Proposition 4.15

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi})$ *be a commutative dendriform T-algebra in the sense of*

$$a \succ_{p,q} b = b \prec_{q,p} a,$$

define

$$a \star_{p,q} b := a \succ_{p,q} b = b \prec_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ *is a Zinbiel T-algebra.*

Corollary

Corollary 4.16

(1) Let (A, R) *be a Rota-Baxter T-algebra. Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *is a pre-Lie T-algebra, where*

$$a *_{p,q} b := R_p(a) \cdot_{p,q} b - b \cdot_{q,p} R_p(a) - \lambda b \cdot_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$*.*

(2) Let (A, R) *be a Rota-Baxter T-algebra such that* $R_p(a)\cdot_{p,q}b = b \cdot_{q,p} R_p(a) + \lambda b \cdot_{q,p} a$*. Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ *is a Zinbiel T-algebra, where*

$$a \star_{p,q} b := R_p(a) \cdot_{p,q} b$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$*.*

Dendriform T-algebra and Zinbiel T-algebra

Lemma 4.17

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ *be a Zinbiel T-algebra. Then*

$$a \star_{p,qt} (b \star_{q,t} c) = b \star_{q,pt} (a \star_{p,t} c)$$

where $a \in A_p$, $b \in A_q$, $c \in A_t$ *and* $p, q, t \in \pi$.

Proposition 4.18

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ *be a Zinbiel T-algebra. Define*

$$a \prec_{p,q} b = b \star_{q,p} a, \quad a \succ_{p,q} b = a \star_{p,q} b,$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\prec_{p,q}\}_{p,q\in\pi}, \{\succ_{p,q}\}_{p,q\in\pi})$ *is a dendriform T-algebra.*

From Zinbiel T-algebras to T-algebras

Proposition 4.19

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ *be a Zinbiel T-algebra. Define*

$$a \diamond_{p,q} b = a \star_{p,q} b + b \star_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\diamond_{p,q}\}_{p,q\in\pi})$ *is a T-algebra.*

Poisson T-algebra

Pre-Poisson algebra was proposed by Aguiar by combining Zinbiel algebra and pre-Lie algebra. We extend it to the T-version.

Definition 4.20

A **Poisson T-algebra** is a triple $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ where $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi})$ is a commutative T-algebra, $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ is a Lie T-algebra and the following condition holds

$$[a, b \cdot_{q,t} c]_{p,qt} = [a, b]_{p,q} \cdot_{pq,t} c + b \cdot_{q,pt} [a, c]_{p,t},$$

for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$.

Pre-Poisson T-algebra

Definition 4.21

A **pre-Poisson T-algebra** is a triple $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ where $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi})$ is a Zinbiel T-algebra, $(\{A_\varphi\}_{\varphi\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ is a pre-Lie T-algebra and the following conditions hold

$$(a *_{p,q} b) \star_{pq,t} c - (b *_{q,p} a) \star_{qp,t} c = a *_{p,qt} (b \star_{q,t} c) - b \star_{q,pt} (a *_{p,t} c),$$
$$(a \star_{p,q} b) *_{pq,t} c + (b \star_{q,p} a) *_{qp,t} c = a \star_{p,qt} (b *_{q,t} c) + b \star_{q,pt} (a *_{p,t} c)$$

for $a \in A_p$, $b \in A_q$, $c \in A_t$ and $p, q, t \in \pi$.

From pre-Poisson T-algebras to Poisson T-algebras

Proposition 4.22

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *be a pre-Poisson* T*-algebra. Define*

$$a \diamond_{p,q} b = a \star_{p,q} b - b \star_{q,p} a, \quad [a,b]_{p,q} = a *_{p,q} b - b *_{q,p} a$$

for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\diamond_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ *is a Poisson* T*-algebra.*

Rota-Baxter Poisson T-algebras

Definition 4.23

A **Rota-Baxter Poisson T-algebra of weight** λ is a triple $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ endowed with a family of linear maps $\{R_\varphi : A_\varphi \longrightarrow A_\varphi\}_{\varphi\in\pi}$ such that
(1) $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi})$ is a Poisson T-algebra,
(2) $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi})$ is a Rota-Baxter T-algebra of weight λ,
(3) $(\{A_\varphi\}_{\varphi\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi})$ is a Rota-Baxter Lie T-algebra of weight λ.
We denote it by $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi}, \lambda)$.

From Rota-Baxter Poisson T-algebras to pre-Poisson T-algebras

Proposition 4.24

Let $(\{A_\varphi\}_{\varphi\in\pi}, \{\mu_{p,q}\}_{p,q\in\pi}, \{[,]_{p,q}\}_{p,q\in\pi}, \{R_\varphi\}_{\varphi\in\pi})$ *be a Rota-Baxter Poisson T-algebra of weight* 0. *Define*

$$a \star_{p,q} b = R_p(a) \cdot_{p,q} b, \quad a *_{p,q} b = [R_p(a), b]_{p,q}$$

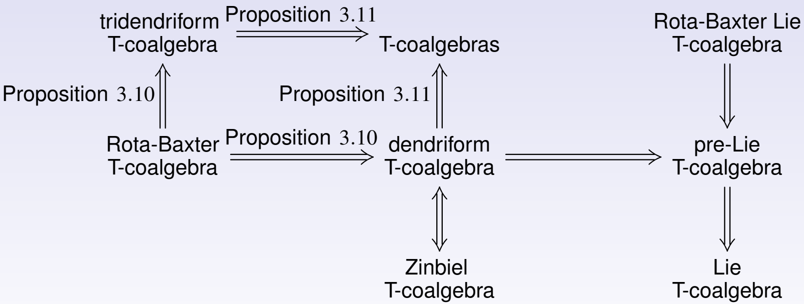
for $a \in A_p$, $b \in A_q$ *and* $p, q \in \pi$. *Then* $(\{A_\varphi\}_{\varphi\in\pi}, \{\star_{p,q}\}_{p,q\in\pi}, \{*_{p,q}\}_{p,q\in\pi})$ *is a pre-Poisson T-algebra.*

Further work

In [4], the authors gave a broad study of representation and module theory of Rota-Baxter algebras. They provided the equivalent characterizations of Rota-Baxter modules by using quasi-idempotency and the ring of Rota-Baxter operators. And in [5], many examples of Rota-Baxter paired modules were obtained from the theory of Hopf algebras. In the forthcoming paper, we will introduce the notion of Rota-Baxter T-modules and investigate its properties, especially pay more attention to the relation between Rota-Baxter operator and Turaev's Hopf group (co)algebras.

Dual cases

For the sake of completeness, we provide the coalgebra version
Throughout we assume that π is commutative semigroup.



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