Root multiplicities for Nichols algebras of diagonal type of rank two

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- Nichols algebra was originally introduced in the late 1970 by W.Nichols in the paper, in which the author attempted to classify certain finite- dimensional Hopf algebra[Comm. Algebra,78].

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- It turned out that Nichols algebras played as a fundamental object in the lifting method to classify finite dimensional pointed Hopf algebras [J.Algebra, 98].
- I. Heckenberger introduced the root system and Weyl groupoid of Nichols algebra of diagonal type[Invent.Math. 2006]. And using the tools of root system and Weyl groupoid he classified the finite-dimensional Nichols algebras of diagonal type over fields of characteristic zero[Compositio.Math. 2007][Algebr.Represent. Theor. 2008],[Rev.Mat.Iberoamericana,Madrid,2007][Adv.Math, 2009].

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- Jing Wang and I. Heckenberger classified the Nichols algebra of diagonal type of rank 2,3 over field of positive characteristic[SIGMA Symmetry Integrability Geom.Methods Appl.2015],[Israel J. Math, 2017]. And Jing Wang classified the case of rank 4 over over field of positive characteristic[J.Algebra. 2021]

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- I. Damiani gave the structures of root vectors of some one-parameter quantum affine [J. Algebra,93],[Publ. Res. Inst. Math. Sci.,12].

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- Roots of form $m\alpha_1 + k\alpha_2, k = \{0, 1\}$, $m \in \mathbb{N}$ were determined in [M.Rosso,Invent.math.98].
- Roots for finite-dimensional Nichols algebras is determined in [I.Heckenberger,Algebra Represent.Theory 08] and [M.Cuntz, I.Heckenberger,J.Pure Appl.Algebra,09]: The roots are real roots with respect to the action of the Weyl groupoid, and their multiplicities are one.

Braided vector space

Let $\Bbbk$ be a field and $\Bbbk^\times = \Bbbk/\{0\}$

Definition

We call a pair (V, c) a braided vector space, if V is a vector space and $c : V \otimes V \to V \otimes V$ is an linear isomorphism of $V \otimes V$, and c satisfies the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Braided vector space of diagonal type

Definition

Let V be a vector space and let $c : V \otimes_{\Bbbk} V \to V \otimes_{\Bbbk} V$ be a linear isomorphism. The pair (V, c) is called a **braided vector space of diagonal type** if there exists a basis $x_1, x_2, \ldots, x_n$ of V, such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \text{ for any } 1 \leq i \leq j \leq n$$

for some $q_{ij} \in \Bbbk^{\times}$. $\boldsymbol{q} = (q_{ij})_{1 \leq i,j \leq n}$ is called the braiding matrix of V.

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- $\mathbb{B}_m$: the braid group generated by $m-1$ standards $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for any } 1 \leq i \leq m-2,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } 1 \leq i+1 < j \leq m-1.$$

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$$\begin{aligned} &\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for any } 1 \leq i \leq m-2, \\ &\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } 1 \leq i+1 < j \leq m-1. \end{aligned}$$

- For $m \geq 2$, let $\rho_m : \Bbbk\mathbb{B}_m \to \mathrm{End}(V^{\otimes m})$ be the representation of $\Bbbk\mathbb{B}_m$ given by $c_i = \rho_m(\sigma_i) = \mathrm{id}_{V^{\otimes i-1}} \otimes c \otimes \mathrm{id}_{V^{\otimes m-i-1}}$.

Braid group and braided symmetrizer

- For any $m \geq 2$, let
$$S_m = \sum_{\sigma \in \mathbb{S}_m} s(\sigma)$$
where $\mathbb{S}_m$ is the symmetric group and $s : \mathbb{S}_m \to \mathbb{B}_m$ is the section map.
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Definition (P. Schauenburg, Comm. Algebra, 1996)

Let (V, c) be a braided vector space. The quotient

$$\mathcal{B}(V) = \Bbbk + V + \bigoplus_{k \geq 2} V^{\otimes k} / \ker(\rho_k(S_k))$$

is called the **Nichols algebra** of V. We say $\mathcal{B}(V)$ if of diagonal type if V is of diagonal type.

Root vector

- Let (V, c) be a braided vector space of diagonal type.
- Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis of V.
- $\mathbb{X}^\times$: the set of non-empty words with letters in X.

Definition

For a Lyndon word $w \in \mathbb{X}^\times$, define the **super-word** $[w] \in T(V)$ inductively given by:

1. $[w] = w$, if $w \in X$, and
2. $[w] = [u][v] - \chi(\deg(u), \deg(v))[v][u]$, and $w = uv$ is the Shirshow decomposition of w.

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Definition

Let $w = v^k$ and v be a Lyndon word. Then $[w]$ is a **root vector** (of $\mathcal{B}(V)$) if $[w] \in \mathcal{B}(V)$ not a linear combination of elements of the form $[v_k]^{m_k}[v_{k-1}]^{m_{k-1}} \cdots [v_1]^{m_1}$, and $[v_1], \ldots, [v_k]$ are super-letters with $w < v_1 < \cdots < v_k$.

Root system

Theorem (V. K. Kharchenko, Algebra Log., 1999)

There exists a subset $L \subseteq \mathbb{X}^\times$ of Lyndon words, such that the elements

$$[v_k]^{m_k} \cdots [v_1]^{m_1}, \quad k \in \mathbb{N}_0,\ v_1, \ldots, v_k \in L,\ v_1 \le v_2 \le \cdots \le v_k,$$
$$0 < m_i < h(v_i) \text{ for any } i \text{ with } h(v_i) \neq 1,$$

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Definition

Let $\Delta_+ = \{deg(u) \mid u \in L\}$ be the **positive roots** of $\mathcal{B}(V)$. And the root system of $\mathcal{B}(V)$ is $\Delta = \Delta_+ \cup -\Delta_+$.

Definition (Multiplicity)

For any $\alpha \in \Delta_+$, the number of elements $u \in L$ with $\deg(u) = \alpha$ is called the **multiplicity** of α.

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- For all $k \in \mathbb{N}_0$, let $b_k = \prod_{j=0}^{k-1}(1 - q^j r)$;

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- For all $k \in \mathbb{N}_0$, let $b_k = \prod_{j=0}^{k-1}(1 - q^j r)$;

- $\mathrm{ad}\, x_1$: the adjoint action of $T(V)$ such that for any $v \in T(V)$
$$\mathrm{ad}\, x_1(v) = x_1 v - \chi(\alpha_1, \deg(v)) v x_1.$$

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Lemma(M.Rosso, Inven.math.98, Lemma 14)

Let $k \in \mathbb{N}$, then $u_k = 0$ in $\mathcal{B}(V)$, if and only if $(k)_q^! b_k = 0$.

The element P_k in $T(V)$

- Write $\hat{u}_k = (k)_q^{!-1} b_k^{-1} u_k$, if $(k)_q^! b_k \neq 0$.

Definition

For all $k \in \mathbb{N}_0$ with $(k)_q^! b_k \neq 0$, let

$$P_k = \sum_{i=0}^{k} (-q_{21})^i q^{i(i-1)/2} \hat{u}_i \hat{u}_{k-i} \in T(V).$$

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Lemma

Let $k \in \mathbb{N}_0$ with $(k)_q^! b_k \neq 0$. Then $P_k = 0$ in $\mathcal{B}(V)$ iff

$$1 + (-r)^k s q^{k(k-1)/2} = 0.$$

The set $\mathbb{J}$

Definition

Let $\mathbb{J} = \mathbb{J}_{q,r,s} \subseteq \mathbb{N}_0$ be such that $j \in \mathbb{J}$ if and only if

$$q^{j(j-1)/2}(-r)^j s + 1 = 0$$

and for any $n \in \mathbb{J}$ with $n < j$ such that

$$\begin{cases} (\frac{j-n}{2})_{q^{n+j-1}r^2} = 0, \text{if } j-n \text{ is even,} \\ (j-n)_{-q^{(n+j-1)/2}r} = 0, \text{if } j-n \text{ is odd.} \end{cases}$$

The set $\mathbb{J}$

Lemma

1. Let $\mathbb{J}_1 \subseteq \mathbb{J}$ such that $j \in \mathbb{J}_1$ iff $q^{j(j-1)/2}(-r)^j s = -1$ and $q^{n+j-1}r^2 \neq 1$, for any $n \in \mathbb{J}$ with $n < j$;
2. Let $\mathbb{J}_2 \subseteq \mathbb{J}$ such that $j \in \mathbb{J}_2$ iff there exists $n \in \mathbb{J}_1$ with $n < j$, such that

$$\begin{cases} q^{n+j-1}r^2 = 1 \text{ and } 2p|(j-n), & \text{if } j-n \text{ is even} \\ q^{(n+j-1)/2}r = -1 \text{ and } p|(j-n), & \text{if } j-n \text{ is odd.} \end{cases}$$

Then $\mathbb{J} = \mathbb{J}_1 \cup \mathbb{J}_2$.

Note that if $\mathrm{char}(\Bbbk) = 0$, then $\mathbb{J} = \mathbb{J}_1$

The case for $\mathrm{char}(\Bbbk) = 0$

- For any $m \in \mathbb{N}_0$, $U_m = \bigoplus_{i=0}^{m} \Bbbk u_i u_{m-i} \subseteq T(V)$

Theorem

Assume that $\mathrm{char}(\Bbbk) = 0$. Let $m \in \mathbb{N}_0$ such that $(m)^!_q b_m \neq 0$. Then the elements $(\mathrm{ad}\, x_1)^{m-j}(P_j)$, $j \in \mathbb{J}_1 \cap [0, m]$ form a basis of $\ker(S_{m+2}) \cap U_m$.

$$U'_n = \bigoplus_{i=0}^{n-1} \Bbbk u_i u_{n-i} \subseteq T(V).$$

Lemma

Assume $\mathrm{char}(\Bbbk) \neq 0$. Let $n \in \mathbb{N}_0$ such that that $(n)^!_q b_n \neq 0$. Suppose that $n \in \mathbb{J}_2$, and that $j_n \in \mathbb{J}_1$ with $j_n < n$ such that $q^{n+j_n-1} r^2 = 1$. Then there exits a unique element $L_n \in U'_n \cap \ker(S_{n+2})$ such that

$$-q_{21}^{-1} d_1(L_n) = (\mathrm{ad}\, x_1)^{n-j_n-1}(P_{j_n}).$$

Theorem

Suppose that $m \in \mathbb{N}_0$ and that $(m)^!_q b_m \neq 0$. The elements

$$(\mathrm{ad}\, x_1)^{m-j}(P_j),\ j \in \mathbb{J}_1 \cap [0, m], \text{and}$$

$$(\mathrm{ad}\, x_1)^{m-n}(L_n), n \in \mathbb{J}_2 \cap [0, m],$$

form a basis of $\ker(S_{m+2}) \cap U_m$.

Multiplicities of $m\alpha_1 + 2\alpha_2$

Corollary

Let $m \in \mathbb{N}_0$ such that $(m)^!_q b_m \neq 0$. Then the multiplicity of $m\alpha_1 + 2\alpha_2$ is

$$m' - \left|\mathbb{J} \cap [0, m]\right|,$$

$$m' = \begin{cases} (m+1)/2 & \text{if } m \text{ is odd}, \\ m/2 & \text{if } m \text{ is even and } q^{m^2/4}(r)^{m/2} q \neq -1, \\ m/2 + 1 & \text{if } m \text{ is even and } q^{m^2/4}(r)^{m/2} q = -1. \end{cases}$$

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Proposition

Assume that $k, m \in \mathbb{N}_0$ with $m \geq k$ such that $(k)^!_q b_k \neq 0$, and $(k+1)_q(1 - q^k r) = 0$. Then the multiplicity of $m\alpha_1 + 2\alpha_2$ is the same as the multiplicity of $(2k - m)\alpha_1 + 2\alpha_2$ of $\mathcal{B}(R_1(V))$.

Corollary

Let $m \in \mathbb{N}_0$.

1. Assume that $m = 2k + 1 \geq 5$ is odd and that $(k+3)^!_q b_{k+3} \neq 0$. Then $m\alpha_1 + 2\alpha_2$ is a root of $\mathcal{B}(V)$.
2. Assume that $m = 2k \geq 8$ and that $(k+4)^!_q b_{k+4} \neq 0$. Then $m\alpha_1 + 2\alpha_2$ is a root of $\mathcal{B}(V)$.
3. Assume that $m \in \{1, 2, 3, 4, 6\}$ and that $(m)^!_q b_m \neq 0$. Then $m\alpha_1 + 2\alpha_2$ is not a root if and only if q, r, s satisfy the conditions given in the following table.

The cases for $m = 1, 2, 3, 4, 6$

$m\alpha_1 + 2\alpha_2$	non-root conditions
$\alpha_1 + 2\alpha_2$	$(1+s)(1-rs) = 0$
$2\alpha_1 + 2\alpha_2$	$(1+s)(1-rs)(1+qr^2s) = 0$
$3\alpha_1 + 2\alpha_2$	$s = -1, (3)_{-qr} = 0$
$4\alpha_1 + 2\alpha_2$	$s = -1, (3)_{-qr} = 0$ or $s = -1, q^3r^2 = -1$ or $rs = 1, (3)_{-q^2r} = 0$
$6\alpha_1 + 2\alpha_2$	$q = 1, s = -1, (3)_{-r} = 0$

Example

Assume that $r = q^{-2},\ s = q,\ q^2 \neq 1$, then the cartan matrix is $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. Apply the results above, we get,
The multiplicity for $\alpha_1 + \alpha_2$ is 1;
The multiplicity for $2\alpha_2$ is 0
The multiplicity for $\alpha_1 + 2\alpha_2$ is 1;
The multiplicity for $2\alpha_1 + 2\alpha_2 = \begin{cases} 1, & q^2 \neq -1 \\ 0, & q^2 = -1 \end{cases}$
(note in $\hat{sl}_2$ is 1)
The multiplicity for $3\alpha_1 + 2\alpha_2$ is 1;
The multiplicity for $m\alpha_1 + 2\alpha_2$, $m \geq 4$, is 0.

Thank you for attension!