Finite dimensional Nichols algebras over Suzuki Hopf algebras

Yuxing Shi
blueponder@foxmail.com

Jiangxi Normal University

August 24, 2021

Outline

1. Background
2. Simple Yetter-Drinfel'd modules over $A_{Nn}^{\mu\lambda}$
3. Finite dimensional Nichols algebras over $A_{Nn}^{\mu\lambda}$
4. Braidings of unsolved cases
5. References

Motivation and related works

Let $\Bbbk$ be an algebraicaly closed field of characteristic 0.

Problem 1 ([Shi20])

How to classify all finite dimensional Hopf algebras over the Suzuki Hopf algebra $A_{Nn}^{\mu\lambda}$?

Related works:

- Masuoka et al. studied the quantum lines over A_{1m}^{++} and A_{1m}^{+-} [CDMM04]
- Fantino et al. [FGM19] classified f. d. Hopf algebras over the dual of dihedral group $\widehat{D}_{2m} \cong \left(A_{1\,2a}^{++}\right)^{\sigma}$ of order $2m$, with $m = 4a \geq 12$.
- [Shi19] classified f. d. Hopf algebras over A_{12}^{+-}(Kac-Paljutkin alg).
- [Shi20][Shi21] investigated f. d. Nichols algebras over simple Yetter-Drinfeld modules of $A_{Nn}^{\mu\lambda}$.

Properties of Suzuki Hopf algebra $A_{Nn}^{\mu\lambda}$

- $A_{Nn}^{\mu\lambda}$ is non-trivial semisimple if $(n,\lambda) \neq (2,+1)$.
- $A_{Nn}^{\mu\lambda}$ is group-theoretical, since

$$1 \to K \to A_{Nn}^{\mu\lambda} \to \Bbbk D_{2n} \to 1,$$

where D_{2n} is dihedral group, $K \cong \begin{cases} C_N \times C_2, & (N,\mu) = (\text{even}, +1), \\ C_{2N}, & \textit{otherwise}. \end{cases}$

- $A_{Nn}^{\mu\lambda}$ is not Morita-equivalent to a group algebra in general. Naidu and Nikshych provided an example A_{12}^{+-} [NN08].

The classification of f. d. non-semismiple Hopf algebras with dual Chevalley property

1. Lifting Method [AS98] introduced by Andruskiewitsch and Schneider in 1998.
2. Let A be a Hopf algebra with coradical H is a Hopf subalgebra. Then $\mathrm{gr}A \cong R\#H$, where $R = \oplus_{n\in\mathbb{N}_0} R(n) \in {}^H_H\mathcal{YD}$ is a braided Hopf algebra.
3. Three steps of Lifting Method:
   - Find all finite dimensional Nichols algebra $\mathfrak{B}(V)$, $V \in {}^H_H\mathcal{YD}$; find an efficient set of relations for $\mathfrak{B}(V)$.
   - If $R = \oplus_{n\in\mathbb{N}_0} R(n)$ is a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$ with $V = R(1)$, decide if $R \simeq \mathfrak{B}(V)$. Here $V = R(1)$ is a braided vector space called the infinitesimal braiding.
   - Given V as in step two, classify all H such that $\mathrm{gr}\,A \simeq \mathfrak{B}(V)\#H$ (lifting).

F. d. Nichols algebras over Semisimple Yetter-Drinfeld modules

- Abelian group type (or Diagonal type):
  Completely classified by Heckenberger [Hec06] [Hec09] (Invent. Math. and Adv. Math.) via the concepts of Weyl groupoid and generalized root system;
- Non-Abelian group type(also called rack type in general):
  Almost finished by Heckenberger and Vendramin [HV17b][HV17a] (J. Eur. Math. Soc.);
- Non-group type: rarely touched.
  I will present a potential example here.

F. d. Nichols algebras over Semisimple Yetter-Drinfeld modules

- Abelian group type (or Diagonal type):
  Completely classified by Heckenberger [Hec06] [Hec09] (Invent. Math. and Adv. Math.) via the concepts of Weyl groupoid and generalized root system;
- Non-Abelian group type(also called rack type in general):
  Almost finished by Heckenberger and Vendramin [HV17b][HV17a] (J. Eur. Math. Soc.);
- Non-group type: rarely touched.
  I will present a potential example here.

Problem 2

Determine f. d. Nichols algebras over indecomposable racks.

Difficult! For example, the classification of pointed (or copinted) Hopf algebras over dihedral group D_{2m} is not completed, mainly since we don't know

$$\dim \mathfrak{B}(\mathcal{D}_{2m+1}, c_q) = ?$$

F. d. Nichols algebras over indecomposable braided vector space

Rank	Dimension	Type	Authors
3	12	$(12)^{\mathbb{S}_3}$	Milinski, Schneider, [MS00]
3	$432(\mathrm{char}\Bbbk = 2)$	$(12)^{\mathbb{S}_3}$	
4	$36\ (\mathrm{char}\Bbbk = 2)$	$(123)^{\mathbb{A}_3}$	[GHV11]
4	$72\ (\mathrm{char}\Bbbk \neq 2)$	$(123)^{\mathbb{A}_4}$	[Gn00]
4	5184	$(123)^{\mathbb{A}_4}$	[HLV12]
6	576	$(12)^{\mathbb{S}_4}$	[MS00]
6	576	$(12)^{\mathbb{S}_4}$	
6	576	$(1234)^{\mathbb{S}_4}$	[AG03]
5	1280	$\mathrm{Aff}(5,2)$	[AG03]
5	1280	$\mathrm{Aff}(5,3)$	[AG03]
7	326592	$\mathrm{Aff}(7,3)$	Graña(2002)
7	326592	$\mathrm{Aff}(7,5)$	Graña(2002)
10	8294400	$(12)^{\mathbb{S}_5}$	Graña(2002)
10	8294400	$(12)^{\mathbb{S}_5}$	Graña(2002)
2	$4m,\ m^2$	$\mathfrak{B}(V_{abe})$	[AG18]

The definition of Suzuki Hopf algebra $A_{Nn}^{\mu\lambda}$

The Suzuki Hopf algebra $A_{Nn}^{\mu\lambda}$ [Suz98] parametrized by integers $N \geq 1$, $n \geq 2$ and μ, $\lambda = \pm 1$, is generated by x_{11}, x_{12}, x_{21}, x_{22} subject to the relations:

$$x_{11}^2 = x_{22}^2, \quad x_{12}^2 = x_{21}^2, \quad \chi_{21}^n = \lambda \chi_{12}^n, \quad \chi_{11}^n = \chi_{22}^n,$$
$$x_{11}^{2N} + \mu x_{12}^{2N} = 1, \quad x_{ij}x_{kl} = 0 \text{ whenever } i+j+k+l \text{ is odd},$$

where we use the following notation for $m \geq 1$,

$$\chi_{11}^m := \overbrace{x_{11}x_{22}x_{11}\ldots\ldots}^{m}, \quad \chi_{22}^m := \overbrace{x_{22}x_{11}x_{22}\ldots\ldots}^{m},$$

$$\chi_{12}^m := \overbrace{x_{12}x_{21}x_{12}\ldots\ldots}^{m}, \quad \chi_{21}^m := \overbrace{x_{21}x_{12}x_{21}\ldots\ldots}^{m}.$$

The coalgebra structure and antipode of $A_{Nn}^{\mu\lambda}$ are given by

$$\Delta(\chi_{ij}^k) = \chi_{i1}^k \otimes \chi_{1j}^k + \chi_{i2}^k \otimes \chi_{2j}^k, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}^{4N-1}.$$

Yetter-Drinfeld modules

Yetter-Drinfeld modules over H:
$(M, \cdot)$ is a left H-module and (M, ρ) is a left H-comodule, subject to:

$$\rho(h \cdot m) = h_{(1)}m_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \forall m \in M, h \in H, \tag{1}$$

where $\rho(m) = m_{(-1)} \otimes m_{(0)}$.

Denote the YD-mod category ${}^H_H\mathcal{YD}$, ${}^H_H\mathcal{YD} \simeq {}_{\mathcal{D}(H^{cop})}\mathcal{M}$

Radford's construction: V is a left H-mod, then $V \boxtimes H \in {}^H_H\mathcal{YD}$.

$$h \cdot (\ell \boxtimes g) = (h_{(2)} \cdot \ell) \boxtimes h_{(1)}gS(h_{(3)}), \tag{2}$$

$$\rho(\ell \boxtimes h) = h_{(1)} \otimes (\ell \boxtimes h_{(2)}), \forall h, g, \in H, \ell \in V. \tag{3}$$

Any simple YD-mod is a submodule for some $V \boxtimes H$.

Simple Yetter-Drinfeld modules over $A_{Nn}^{\mu\lambda}$ with n even

Theorem 3 ([Shi20])

A full set of non-isomorphic simple Yetter-Drinfeld modules over $A_{N\,2n}^{\mu\lambda}$ is given by the following list.

1. There are $8N^2$ non-isomorphic Yetter-Drinfeld modules of dimenion one:
   - $\mathscr{A}_{iik,p}^{s}$, $i, p \in \mathbb{Z}_2$, $s \in \overline{1, N}$, $k \in \overline{0, N-1}$;
   - $\bar{\mathscr{A}}_{i\,i+1\,k,p}^{s}$, $i, p \in \mathbb{Z}_2$, $s \in \overline{1, N}$, $k \in \overline{0, N-1}$.
2. There are $2N^2(4n^2 - 1)$ non-isomorphic Yetter-Drinfeld modules of dimension two: $\mathscr{B}_{01k}^{s}$, $\mathscr{C}_{ijk,p}^{st}$, $\mathscr{D}_{jk,p}^{st}$, $\mathscr{E}_{jk,p}^{st}$, $\mathscr{G}_{jk,p}^{st}$, $\mathscr{H}_{jk,p}^{st}$, $\mathscr{P}_{ijk,p}^{st}$.
3. There are $8N^2$ non-isomorphic Yetter-Drinfeld modules of dimension $2n$:
   - $\mathscr{I}_{pjk}^{s}$, $j = 2$ or 4, $k \in \overline{0, N-1}$, $p \in \mathbb{Z}_2$, $s \in \overline{1, N}$;
   - $\mathscr{K}_{jk,p}^{s}$, $j = \begin{cases} 1 \text{ or } 3, & \text{if } \lambda = -1, \\ 2 \text{ or } 4, & \text{if } \lambda = 1, \end{cases}$ $k \in \overline{0, N-1}$, $p \in \mathbb{Z}_2$, $s \in \overline{1, N}$.

Simple Yetter-Drinfeld modules over $A_{Nn}^{\mu\lambda}$ with n odd

Theorem 4 ([Shi21])

A complete set of simple Yetter-Drinfeld modules over the Suzuki Hopf algebra $A_{N\,2n+1}^{\mu\lambda}$ is given as follows.

1. There are $8N^2$ pairwise non-isomorphic Yetter-Drinfeld modules of dimenion one:
   - $\boldsymbol{A}_{k,p}^{s}$, $s \in \overline{1,N}$, $k \in \overline{0,2N-1}$, $p \in \mathbb{Z}_2$;
   - $\boldsymbol{B}_{k,p}^{s}$, $s \in \overline{1,N}$, $k \in \overline{0,2N-1}$, $p \in \mathbb{Z}_2$.
2. There are $8N^2n(n+1)$ pairwise non-isomorphic Yetter-Drinfeld modules of dimension two: $\boldsymbol{C}_{jk,p}^{st}$, $\boldsymbol{D}_{jk,p}^{st}$, $\boldsymbol{E}_{jk,p}^{s}$, $\boldsymbol{F}_{k,p}^{st}$, $\boldsymbol{G}_{k,p}^{st}$, $\boldsymbol{H}_{jk,p}^{st}$, $\boldsymbol{I}_{jk,p}^{st}$,
3. There are $8N^2$ pairwise non-isomorphic Yetter-Drinfeld modules of dimension $2n+1$:
   - $\boldsymbol{L}_{k,pq}^{s}$, $s \in \overline{1,N}$, $k \in \overline{0,N-1}$, $p,q \in \mathbb{Z}_2$;
   - $\boldsymbol{N}_{k,pq}^{s}$, $s \in \overline{1,N}$, $k \in \overline{0,N-1}$, $p,q \in \mathbb{Z}_2$.

The definition of Nichols algebra

Definition 5

Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $R = \bigoplus_{n\geq 0} R(n) \in {}^H_H\mathcal{YD}$ is called the Nichols algebra of V if

(i) $\mathbb{K} \simeq R(0)$, $V \simeq R(1) \in {}^H_H\mathcal{YD}$,

(ii) $R(1) = \mathcal{P}(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$.

(iii) R is generated as an algebra by $R(1)$.

In this case, R is denoted by $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$.

The Nichols algebra $\mathfrak{B}(V)$ is completely determined by the braiding.

$$\mathfrak{B}(V) = \Bbbk \oplus V \oplus \bigoplus_{n=2}^{\infty} V^{\otimes n} / \ker \mathfrak{S}_n = T(V)/\ker \mathfrak{S},$$

where $\mathfrak{S}_{n,1} \in \mathrm{End}_{\mathbb{K}}\left(V^{\otimes(n+1)}\right)$, $\mathfrak{S}_n \in \mathrm{End}_{\mathbb{K}}\left(V^{\otimes n}\right)$,

$$\mathfrak{S}_{n,1} := \mathrm{id} + c_n + c_n c_{n-1} + \cdots + c_n c_{n-1} \cdots c_1 = \mathrm{id} + c_n \mathfrak{S}_{n-1,1},$$

$$\mathfrak{S}_1 := \mathrm{id}, \quad \mathfrak{S}_2 := \mathrm{id} + c, \quad \mathfrak{S}_n := (\mathfrak{S}_{n-1} \otimes \mathrm{id})\mathfrak{S}_{n-1,1}.$$

Let $\mathfrak{B}(V)$ be a Nichols algebra of diagonal type, i.e., V is of diagonal type with braiding matrix (q_{ij}), $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$. For diagonal $\mathfrak{B}(V)$ with braiding matrix $(q_{ij})_{n\times n}$ we associate a generalized Dynkin diagram: this is a graph with n vertices, where the i-th vertex is labeled with q_{ii} for all $1 \le i \le n$; further, if $q_{ij}q_{ji} \neq 1$, then there is an edge between the i-th and j-th vertex labeled with $q_{ij}q_{ji}$.

Figure: $q_{ij}q_{ji} = 1$ resp. $q_{ij}q_{ji} \neq 1$

Recall that V resp. $\mathfrak{B}(V)$ is called of Cartan type, if there is a generalized Cartan matrix (a_{ij}) such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$.

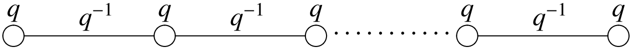

Figure: generalized dynkin diagram for Cartan type A_n

Theorem 6 ([Shi20, Shi21])

Let M be a simple Yetter-Drinfeld module over $A_{Nn}^{\mu\lambda}$. If $\mathfrak{B}(M)$ is of diagonal type and $\dim \mathfrak{B}(M) < \infty$, then $\mathfrak{B}(M)$ can be classified as follows.

1. Cartan type A_1;
2. Cartan type $A_1 \times A_1$;
3. Cartan type A_2, $\overset{q}{\circ}\overset{q^{-1}}{\rule[0.5ex]{4em}{0.4pt}}\overset{q}{\circ}$, $q \neq 1$;
4. Cartan type $A_2 \times A_2$, $\overset{-1}{\circ}\overset{-1}{\rule[0.5ex]{4em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{-1}{\rule[0.5ex]{4em}{0.4pt}}\overset{-1}{\circ}$;
5. Super type $\mathbf{A}_2(q;\mathbb{I}_2)$, $\overset{-1}{\circ}\overset{q}{\rule[0.5ex]{4em}{0.4pt}}\overset{-1}{\circ}$, $q \neq \pm 1$;
6. The Nichols algebra $\mathfrak{ufo}(8)$, $\overset{-\zeta^2}{\circ}\overset{\zeta}{\rule[0.5ex]{4em}{0.4pt}}\overset{-\zeta^2}{\circ}$, $\zeta \in \mathbb{G}_{12}$.

F.d. Nichols algebras of non-diagonal type:

1. Dimension 12
2. Dimension 64
3. $\dim \mathfrak{B}(V_{abe}) = \begin{cases} 4m, & b^2 \neq ae, b = -1, ae \in \mathbb{G}_m, \\ m^2, & b^2 \neq ae, ae = 1, b \in \mathbb{G}_m \text{ for } m \geq 2, \\ \infty, & ae \neq b^2 = (ae)^{-1}, b \in \mathbb{G}_m \text{ for } m \geq 3, \\ \infty, & b \notin \mathbb{G}_m \text{ for } m \geq 2, \\ \text{unknown}, & \textit{otherwise}. \end{cases}$

Unsolved cases for Nichols algebras over simple YD-modules:

1. The unknown cases for $\mathfrak{B}(V_{abe})$;
2. The unknown cases for $\mathfrak{B}\left(\mathscr{K}^s_{jk,p}\right)$, $n \geq 2$;
3. The unknown cases for $\mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)$;
4. The unknown cases for $\mathfrak{B}\left(\boldsymbol{N}^s_{k,pq}\right)$.

Those unsolved cases were proposed as open problems.

Let $V_{abe} = \Bbbk v_1 \oplus \Bbbk v_2$ be a vector space with a braiding given by

$$c(v_1 \otimes v_1) = a v_2 \otimes v_2, \qquad c(v_1 \otimes v_2) = b v_1 \otimes v_2,$$
$$c(v_2 \otimes v_1) = b v_2 \otimes v_1, \qquad c(v_2 \otimes v_2) = e v_1 \otimes v_1,$$

then the braided vector space (V_{abe}, c) is of type V_{abe}. The braided vector space $V_{abe} \cong V_{ae\,b\,1}$ via $v_1 \mapsto \sqrt{e} v_1$, $v_2 \mapsto v_2$.
When $ae = b^2$, then V_{abe} is of diagonal type and

$$\dim \mathfrak{B}(V_{abe}) = \begin{cases} 4, & b = -1, (\text{Cartan type } A_1 \times A_1), \\ 27, & b^3 = 1 \neq b, (\text{Cartan type } A_2), \\ \infty, & \text{otherwise.} \end{cases}$$

When $ae \neq b^2$, according to [AG18] and [Shi20, Shi21], we have

$$\dim \mathfrak{B}(V_{abe}) = \begin{cases} 4m, & b = -1, ae \in \mathbb{G}_m, \\ m^2, & ae = 1, b \in \mathbb{G}_m \text{ for } m \geq 2, \\ \infty, & b^2 = (ae)^{-1}, b \in \mathbb{G}_m \text{ for } m \geq 3, \\ \infty, & b \notin \mathbb{G}_m \text{ for } m \geq 2, \\ \text{unknown}, & \textit{otherwise}, \end{cases} \tag{4}$$

Prove $\dim \mathfrak{B}(V_{abe}) = m^2$ in case $ae = 1, b \in \mathbb{G}_m$, $m \geq 2$

Proof sketch 1: Let $V = \{1, 2\}$, $V^n = \{i_1 i_2 \cdots i_n \mid i_j \in V, 1 \leq j \leq n\}$. There is an action of $\mathbb{S}_n$ on V^n as follows

$$s_k \cdot (i_1 \cdots i_{k-1} pq i_{k+2} \cdots i_n) = \begin{cases} i_1 \cdots i_{k-1} pq i_{k+2} \cdots i_n, & pq = 12 \text{ or } 21, \\ i_1 \cdots i_{k-1} 22 i_{k+2} \cdots i_n, & pq = 11, \\ i_1 \cdots i_{k-1} 11 i_{k+2} \cdots i_n, & pq = 22. \end{cases}$$

Theorem 7 (The orders of orbits of V^n forms a Pascal's triangle)

$$V^{2n} = \mathcal{O}\left(2^{2n}\right) \oplus \bigoplus_{k=1}^{n} \left[\mathcal{O}\left(2^{2(n-k)}(21)^k\right) \oplus \mathcal{O}\left(1^{2(n-k)}(12)^k\right)\right],$$

$$V^{2n+1} = \bigoplus_{k=0}^{n} \left[\mathcal{O}\left(2^{2(n-k)+1}(12)^k\right) \oplus \mathcal{O}\left(1^{2(n-k)+1}(21)^k\right)\right],$$

$$\left|\mathcal{O}\left(2^{2(n-k)}(12)^k\right)\right| = \binom{2n}{n-k}, \quad \left|\mathcal{O}\left(2^{2(n-k)+1}(12)^k\right)\right| = \binom{2n+1}{n-k}.$$

Proof sketch 2: Since $V_{abe} \cong V_{ae\,b\,1}$, we suppose $a = e = 1$.
$\mathfrak{S}_2(v_1^2) = v_1^2 + v_2^2 = \mathfrak{S}_2(v_2^2)$, so $\mathfrak{S}_n(v_x) = \mathfrak{S}_n(v_y)$ for any $y \in \mathcal{O}(x) \subset N^n$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & (0)_b^! & & & & & & \\
 & & & & & (1)_b^!(0)_b^! & & (0)_b^!(1)_b^! & & & & & \\
 & & & & (2)_b^!(0)_b^! & & (1)_b^!(1)_b^! & & (0)_b^!(2)_b^! & & & & \\
 & & & (3)_b^!(0)_b^! & & (2)_b^!(1)_b^! & & (1)_b^!(2)_b^! & & (0)_b^!(3)_b^! & & & \\
 & & (4)_b^!(0)_b^! & & (3)_b^!(1)_b^! & & (2)_b^!(2)_b^! & & (1)_b^!(3)_b^! & & (0)_b^!(4)_b^! & & \\
 & (5)_b^!(0)_b^! & & (4)_b^!(1)_b^! & & (3)_b^!(2)_b^! & & (2)_b^!(3)_b^! & & (1)_b^!(4)_b^! & & (0)_b^!(5)_b^! & \\
(6)_b^!(0)_b^! & & (5)_b^!(1)_b^! & & (4)_b^!(2)_b^! & & (3)_b^!(3)_b^! & & (2)_b^!(4)_b^! & & (1)_b^!(5)_b^! & & (0)_b^!(6)_b^! \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \quad \cdots
\end{array}
$$

Figure: coefficients of the braided symmetrizer's action on the basis of $\mathfrak{B}(V_{1b1})$

Proposition 8 ([Shi20][Shi21])

Suppose $ae \neq b^2 = (ae)^{-1}$ and $b \in \mathbb{G}_m$ for $m \geq 3$, then

$$\dim \mathfrak{B}(V_{abe}) = \infty.$$

Proof sketch: D_{4n} is dihedral group of order $4n$. By Masuoka's result,

$$\begin{aligned} &{}^{A_{1n}^{++}}_{A_{1n}^{++}}\mathcal{YD} \longrightarrow {}^{D_{4n}}_{D_{4n}}\mathcal{YD} \\ &\mathfrak{B}(V_{abe}) \mapsto \mathfrak{B}(M), \text{diagonal type, it's dimension is 4 or } \infty \end{aligned}$$

$V_{abe} \in {}^{A_{1n}^{++}}_{A_{1n}^{++}}\mathcal{YD}$ with $ae \neq b^2 = (ae)^{-1}$. Only need to prove that b can be any primitive m-th root of unity.

Nichols algebras of dimension 64: the case $\mathfrak{B}\left(\mathscr{I}^s_{pjk}\right)$

Proposition 9

- When $n > 2$, $\mathfrak{B}\left(\mathscr{I}^s_{pjk}\right)$ is of rack type D, so $\dim \mathfrak{B}\left(\mathscr{I}^s_{pjk}\right) = \infty$.
- When $n = 2$, $\dim \mathfrak{B}\left(\mathscr{I}^s_{pjk}\right) = \begin{cases} 64, & (-1)^p \omega^{4k(2s+1)} = -1, \\ \infty, & \text{otherwise.} \end{cases}$

Remark 10

- When $n = 2$, $\mathfrak{B}\left(\mathscr{I}^s_{pjk}\right) \simeq \mathfrak{B}(\mathcal{D}_4, c_q)$, where $\mathcal{D}_4$ is dihedral rack. According to [HV17b], $\dim \mathfrak{B}(\mathcal{D}_4, c_q) < \infty$ implies $\dim \mathfrak{B}(\mathcal{D}_4, c_q) = 64$.
- A rack is a pair $(X, \triangleright)$, $\triangleright\colon X \times X \to X$ such that $\varphi_x = x \triangleright \_\_$ is bijective, and $x \triangleright (y \triangleright z) = (x \triangleright) \triangleright (x \triangleright z)$ for $\forall x, y, z \in X$.
- Dihedral rack $\mathcal{D}_n = \{d_i \mid i \in \mathbb{Z}_n\}$, $d_i \triangleright d_j = d_{2i-j}$.
- A rack X is of type D if there exist a subrack $Y = S \sqcup T$ such that $s \triangleright (t \triangleright (s \triangleright t)) \neq t, \quad$ for some $s \in S, t \in T$.

Suppose $n = 2$ and $(-1)^p\omega^{4k(2s+1)} = -1$, and denote $a = \omega^{8kns} = \omega^{-8kn(s+1)}$, $b = \omega^{2njN} = \pm 1$. Then the braiding of $\mathscr{I}_{pjk}^{s}$ is given by

$$
\begin{aligned}
c(w_1 \otimes w_1) &= -w_1 \otimes w_1, & c(w_1 \otimes w_2) &= -bw_2 \otimes w_1,\\
c(w_2 \otimes w_1) &= -bw_1 \otimes w_2, & c(w_2 \otimes w_2) &= -w_2 \otimes w_2,\\
c(w_1 \otimes m_1) &= am_2 \otimes w_1, & c(w_1 \otimes m_2) &= a^{-1}m_1 \otimes w_1,\\
c(w_2 \otimes m_1) &= \lambda abm_2 \otimes w_2, & c(w_2 \otimes m_2) &= \lambda a^{-1}bm_1 \otimes w_2,\\
c(m_1 \otimes m_1) &= -m_1 \otimes m_1, & c(m_1 \otimes m_2) &= -\lambda bm_2 \otimes m_1,\\
c(m_2 \otimes m_1) &= -\lambda bm_1 \otimes m_2, & c(m_2 \otimes m_2) &= -m_2 \otimes m_2,\\
c(m_1 \otimes w_1) &= aw_2 \otimes m_1, & c(m_1 \otimes w_2) &= a^{-1}w_1 \otimes m_1,\\
c(m_2 \otimes w_1) &= abw_2 \otimes m_2, & c(m_2 \otimes w_2) &= a^{-1}bw_1 \otimes m_2.
\end{aligned}
$$

$W_1 = \Bbbk w_1 \oplus \Bbbk w_2$ is Cartan type $A_1 \times A_1$,
$W_2 = \Bbbk m_1 \oplus \Bbbk m_2$ is Cartan type $A_1 \times A_1$,
$\mathscr{I}_{pjk}^{s} = W_1 \oplus W_2$ as braided vector spaces.

The relations of the 64-dimensional Nichols algebra $\mathfrak{B}\left(\mathscr{I}_{pjk}^{s}\right)$ is given by

$$w_1w_2 + bw_2w_1 = 0, \quad w_i^2 = 0, \quad i \in \overline{1,2},$$
$$m_1m_2 + bm_2m_1 = 0, \quad m_i^2 = 0, \quad i \in \overline{1,2},$$
$$w_1m_1 - am_2w_1 + a^2bw_2m_2 - am_1w_2 = 0,$$
$$w_1m_2 - a^{-1}m_1w_1 + w_2m_1 - abm_2w_2 = 0;$$

If $b = -1$, $x^2 = -a^2$, then

$$[(w_1 + xw_2)(m_1 + xm_2)]^2 + [(m_1 + xm_2)(w_1 + xw_2)]^2 = 0;$$

If $b = 1$, then

$$(w_1 + xw_2)(m_1 + xm_2) - \frac{x}{a}(m_1 + xm_2)(w_1 + xw_2) = 0, \quad x^2 = a^2,$$
$$[(w_1 + xw_2)(ym_1 + zm_2)]^2 + [(ym_1 + zm_2)(w_1 + xw_2)]^2 = 0, \forall x, y, z \in \mathbb{k}.$$

In this case, it's of Cartan type $A_2 \times A_2$, see Milinski-Schneider [MS00].

Nichols algebras of dimension 64: the case $\mathfrak{B}\left(\mathscr{K}_{jk,p}^{s}\right)$

Proposition 11

Suppose $(-1)^p\bar{\mu}^s\omega^{8kns} = -1$, $\lambda = 1$, then $\dim\mathfrak{B}\left(\mathscr{K}_{jk,p}^{s}\right) = 64$ for $n = 2$.

Let $a = \bar{\mu}^2\omega^{16kn}$, $b = \omega^{4jN} = \pm 1$. The braiding is given by

$$
\begin{aligned}
c(w_1\otimes w_1) &= -w_1\otimes w_1, & c(w_1\otimes w_2) &= -w_2\otimes w_1,\\
c(w_1\otimes w_3) &= -w_3\otimes w_1, & c(w_1\otimes w_4) &= -w_4\otimes w_1,\\
c(w_2\otimes w_1) &= -a^{-1}w_3\otimes w_4, & c(w_2\otimes w_2) &= -a^{-1}bw_4\otimes w_4,\\
c(w_2\otimes w_3) &= -bw_1\otimes w_4, & c(w_2\otimes w_4) &= -w_2\otimes w_4,\\
c(w_3\otimes w_1) &= -w_1\otimes w_3, & c(w_3\otimes w_2) &= -w_2\otimes w_3,\\
c(w_3\otimes w_3) &= -w_3\otimes w_3, & c(w_3\otimes w_4) &= -w_4\otimes w_3,\\
c(w_4\otimes w_1) &= -bw_3\otimes w_2, & c(w_4\otimes w_2) &= -w_4\otimes w_2,\\
c(w_4\otimes w_3) &= -aw_1\otimes w_2, & c(w_4\otimes w_4) &= -abw_2\otimes w_2.
\end{aligned}
$$

$W_1 = \Bbbk w_1\oplus\Bbbk w_3$ is Cartan type $A_1\times A_1$, $W_2 = \Bbbk w_2\oplus\Bbbk w_4\cong V_{1\,-1\,1}$,
$\mathscr{K}_{jk,p}^{s} = W_1\oplus W_2$ is non-group type (Not checked strictly!)

Nichols algebras of dimension 64: the case $\mathfrak{B}\left(\mathscr{K}_{jk,p}^{s}\right)$

The relations of the Nichols algebra $\mathfrak{B}\left(\mathscr{K}_{jk,p}^{s}\right)$ of dimension 64:

$$\begin{aligned} w_1^2 = 0, \quad w_3^2 = 0, \quad w_1w_3 + w_3w_1 &= 0, \\ w_2w_4 = 0, \quad w_4w_2 = 0, \quad w_2^2 + \frac{b}{a}w_4^2 &= 0, \\ w_1w_2 + w_2w_1 + a^{-1}w_3w_4 + a^{-1}w_4w_3 &= 0, \\ w_1w_4 + w_4w_1 + bw_3w_2 + bw_2w_3 &= 0, \\ w_1w_2w_3w_4 &= aw_2w_1w_2w_1. \end{aligned}$$

$$[(w_1 + xw_3)(yw_2 \pm w_4)]^2 + [(yw_2 \pm w_4)(w_1 + xw_3)]^2 = 0,$$
$$[(xw_1 + w_3)(yw_2 \pm w_4)]^2 + [(yw_2 \pm w_4)(xw_1 + w_3)]^2 = 0,$$

for any $x \in \mathbb{k}$, $y^2 = ab$.

Lemma 12

When $n=1$, $p=1$, $k=0$, then $\dim \mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)=12$. It is generated by m_1, m_2, m_3 and with relations

$$m_3m_2 = m_1m_3 - m_2m_1, \quad m_2m_3 = -m_1m_2 + m_3m_1, \quad m_i^2 = 0 \quad \forall i \in \overline{1,3}. \tag{5}$$

When $n=1$, the braiding of $\mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)$ is given by

$$\begin{aligned}
c(m_1\otimes m_1) &= (-1)^p\omega^{2k(2n+1)(2s-1)}m_1\otimes m_1,\\
c(m_1\otimes m_2) &= \omega^{4k(2n+1)(s-1)}m_3\otimes m_1,\\
c(m_1\otimes m_3) &= \omega^{4k(2n+1)s}m_2\otimes m_1,\\
c(m_2\otimes m_1) &= (-1)^p\omega^{2k(2n+1)[1+2(s-2)]}m_3\otimes m_2,\\
c(m_2\otimes m_2) &= (-1)^p\omega^{2k(2n+1)(2s-1)}m_2\otimes m_2,\\
c(m_2\otimes m_3) &= (-1)^p\omega^{2k(2n+1)(1+2s)}m_1\otimes m_2,\\
c(m_3\otimes m_1) &= \omega^{4k(2n+1)(s-1)}m_2\otimes m_3,\\
c(m_3\otimes m_2) &= \omega^{4k(2n+1)s}m_1\otimes m_3,\\
c(m_3\otimes m_3) &= (-1)^p\omega^{2k(2n+1)(2s-1)}m_3\otimes m_3.
\end{aligned}$$

Lemma 13

When $n = 1 = \mu$, $k = 0$, $q = 1$, then $\dim \mathfrak{B}\left(\boldsymbol{N}^s_{k,pq}\right) = 12$. It is generated by w_1, w_2, w_3 and with relations:

$$w_1^2 + (-1)^p w_2 w_3 + (-1)^p w_3 w_2 = 0, \quad w_2^2 = 0, \tag{6}$$

$$w_1 w_2 + \lambda(-1)^p w_3^2 + w_2 w_1 = 0, \quad w_1 w_3 = 0, \quad w_3 w_1 = 0. \tag{7}$$

When $n = 1$, the braiding of $\mathfrak{B}\left(\boldsymbol{N}^s_{k,pq}\right)$ is given by

$$\begin{aligned}
c(w_1 \otimes w_1) &= (-1)^{p+q} B^{2s-1} w_2 \otimes w_3, & c(w_1 \otimes w_2) &= \lambda(-1)^{p+q} B^{2s-1} w_3 \otimes w_3,\\
c(w_1 \otimes w_3) &= (-1)^{q} B^{2s+2} w_1 \otimes w_3, & c(w_2 \otimes w_1) &= (-1)^{q} B^{2s+2} w_1 \otimes w_2,\\
c(w_2 \otimes w_2) &= (-1)^{q} B^{2s+2} w_2 \otimes w_2, & c(w_2 \otimes w_3) &= (-1)^{q} B^{2s+2} w_3 \otimes w_2,\\
c(w_3 \otimes w_1) &= (-1)^{q} B^{2s+2} w_3 \otimes w_1, & c(w_3 \otimes w_2) &= (-1)^{p+q} B^{2s+5} w_1 \otimes w_1,\\
c(w_3 \otimes w_3) &= \lambda(-1)^{p+q} B^{2s+5} w_2 \otimes w_1, & B &= \bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}.
\end{aligned}$$

Theorem 14

Finite dimensional Nichols algebras over $A_{13}^{+\lambda}$, associated with simple Yetter-Drinfeld modules, can be classified as follows.

1. $\dim \mathfrak{B}\left(\mathscr{B}_{k,p}^{s}\right) = \begin{cases} 2, & \text{if } \lambda = 1, s = 1, k = 0, p = 1, \\ 4, & \text{if } \lambda = -1, s = 1, k = 0, p \in \mathbb{Z}_2. \end{cases}$
2. $\dim \mathfrak{B}\left(\mathscr{G}_{k,p}^{st}\right) = 4$ in case $\lambda = 1 = s$, $t = 0$, $(p, k) = (0, 1)$ or $(1, 0)$.
3. $\dim \mathfrak{B}\left(\mathscr{L}_{k,pq}^{s}\right) = 12$ in case $s = p = 1$, $k = 0$, $q \in \mathbb{Z}_2$.
4. $\dim \mathfrak{B}\left(\mathscr{N}_{k,pq}^{s}\right) = 12$ in case $s = q = 1$, $k = 0$, $p \in \mathbb{Z}_2$.

The braiding of $\mathfrak{B}\left(\mathscr{K}_{jk,p}^{s}\right)$

Let

$$b+2a-2=2nr+d,\quad r\in\mathbb{N},\quad 0\le d\le 2n-1,$$
$$2n+1-b+2a-2=2ne+f,\quad e\in\mathbb{N},\quad 0\le f\le 2n-1,$$

then the braiding of $\mathscr{K}_{jk,p}^{s}$ is given by

$$c(w_a\otimes w_b)$$
$$=\begin{cases}(-1)^p\left(\bar{\mu}\omega^{8nk}\right)^s w_b\otimes w_1, & a=1,\\ (-1)^p\lambda^r\bar{\mu}^{s+n(r-2)}\omega^{2n(r-2)(4nk+jN)+8nks}w_{2n}\otimes w_{2n-a+2}, & a>1, d=0, 2\mid(a+b),\\ (-1)^p\lambda^{r+1}\bar{\mu}^{s+n(r-1)}\omega^{2n(r-1)(4nk+jN)+8nks}w_d\otimes w_{2n-a+2}, & a>1, d>0, 2\mid(a+b),\\ (-1)^p\lambda^e\left(\bar{\mu}\omega^{8nk}\right)^{s-ne-2+2a}\omega^{-2jNne}w_1\otimes w_{2n-a+2}, & a>1, f=0, 2\nmid(a+b),\\ \frac{(-1)^p\lambda^{e+1}\left(\bar{\mu}\omega^{8nk}\right)^{s-n(e+1)-2+2a}}{\omega^{2jNn(e+1)}}w_{2n+1-f}\otimes w_{2n-a+2}, & a>1, f>0, 2\nmid(a+b).\end{cases}$$

The braiding of $\mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)$

The braiding of $\mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)$ can be described as follows.

1. When $2 \nmid (a+b)$, let $b+(2a-1) = d(2n+1)+r$, $d \in \mathbb{N}$, $r \in \overline{0, 2n}$. Then

$$c(m_a \otimes m_b) = \begin{cases} \omega^{4k(2n+1)(s-a)} m_{b+2a-1} \otimes m_a, & d = 0, \\ \omega^{4k(2n+1)(s-a)} m_{2n+1} \otimes m_a, & d = 1, r = 0 \\ (-1)^p \omega^{2k(2n+1)[2(s-a+r)-1]} m_{2n-r+2} \otimes m_a, & d = 1, r \neq 0, \\ (-1)^p \omega^{2k(2n+1)[2(s-a)+4n+1]} m_1 \otimes m_a, & d = 2, r = 0, \\ \omega^{4k(2n+1)[(s-a)+2n+1]} m_r \otimes m_a, & d = 2, r \neq 0. \end{cases}$$

2. When $2 \mid (a+b)$, let $2a-1 = b+d(2n+1)+r$, $d \in \mathbb{N}$, $r \in \overline{0, 2n}$. Then

$$c(m_a \otimes m_b) = \begin{cases} \omega^{4k(2n+1)(s+a-1)} m_{b-(2a-1)} \otimes m_a, & 2a-1 < b, \\ (-1)^p \omega^{2k(2n+1)[2(s-a+b)-1]} m_{2a-b} \otimes m_a, & d = 0, \\ \omega^{4k(2n+1)[s-a+b+r]} m_{2n+1-r} \otimes m_a, & d = 1. \end{cases}$$

$\mathfrak{B}\left(\boldsymbol{L}^s_{k,pq}\right)$ is of rack type. When $n = 1$, the associated rack is the dihedral rack $\mathcal{D}_3$.

The braiding of $\mathfrak{B}\left(\boldsymbol{N}^{s}_{k,pq}\right)$

Define $R^{\gamma}: \boldsymbol{N}^{s}_{k,pq}\otimes \boldsymbol{N}^{s}_{k,pq} \longrightarrow \boldsymbol{N}^{s}_{k,pq}\otimes \boldsymbol{N}^{s}_{k,pq}$ such that $R^{\gamma}(w_{\alpha}\otimes w_{\beta}) =$

$$\begin{cases} w_{\beta+\gamma}\otimes w_{2n-\alpha+2}, & \beta+\gamma\leq 2n+1,\\ (-1)^{p}\lambda\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}w_{2n+1}\otimes w_{2n-\alpha+2}, & \beta+\gamma = 2n+2,\\ (-1)^{p}\lambda\left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{2(\gamma+\beta)-4n-3} w_{4n+3-\gamma-\beta}\otimes w_{2n-\alpha+2}, & \beta+\gamma\in\overline{2n+3,4n+2},\\ \lambda\left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{4n+2} w_{1+\beta+\gamma-(4n+3)}\otimes w_{2n-\alpha+2}, & \beta+\gamma\geq 4n+3,\end{cases}$$

and $L^{\gamma}: \boldsymbol{N}^{s}_{k,pq}\otimes \boldsymbol{N}^{s}_{k,pq} \longrightarrow \boldsymbol{N}^{s}_{k,pq}\otimes \boldsymbol{N}^{s}_{k,pq}$ such that

$$L^{\gamma}(w_{\alpha}\otimes w_{\beta}) = \begin{cases} \left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{2\gamma} w_{\beta-\gamma}\otimes w_{2n-\alpha+2}, & \gamma<\beta,\\ (-1)^{p}\left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{2\beta-1} w_{\gamma-\beta+1}\otimes w_{2n-\alpha+2}, & \gamma\in\overline{\beta,\beta+2n},\\ \lambda\left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{2\beta} w_{2n+1}\otimes w_{2n-\alpha+2}, & \gamma=\beta+2n+1,\\ \lambda\left[\bar{\mu}^{\frac{1}{2}}\omega^{2k(2n+1)}\right]^{2(\gamma-2n-1)} w_{4n+2+\beta-\gamma}\otimes w_{2n-\alpha+2}, & \gamma>\beta+2n+1.\end{cases}$$

The braiding of $\mathfrak{B}\left(\boldsymbol{N}_{k,pq}^{s}\right)$

The braiding of $\mathfrak{B}\left(\boldsymbol{N}_{k,pq}^{s}\right)$ can be described as follows.

1. When $\alpha = n+1$, then
$$c(w_\alpha \otimes w_\beta) = (-1)^q \left[\bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}\right]^{2(\alpha+s-1)} w_\beta \otimes w_{2n-\alpha+2};$$
2. When $\alpha < n+1$, $\alpha+\beta \equiv 0 \mod 2$, then
$$c(w_\alpha \otimes w_\beta) = (-1)^q \left[\bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}\right]^{2(2\alpha+s-n-2)} L^{2(n-\alpha+1)}(w_\alpha \otimes w_\beta);$$
3. When $\alpha > n+1$, $\alpha+\beta \equiv 1 \mod 2$, then
$$c(w_\alpha \otimes w_\beta) = (-1)^q \left[\bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}\right]^{2(s+n)} L^{2(\alpha-1-n)}(w_\alpha \otimes w_\beta);$$
4. When $\alpha < n+1$, $\alpha+\beta \equiv 1 \mod 2$, then
$$c(w_\alpha \otimes w_\beta) = (-1)^q \left[\bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}\right]^{2(2\alpha+s-n-2)} R^{2(n-\alpha+1)}(w_\alpha \otimes w_\beta);$$
5. When $\alpha > n+1$, $\alpha+\beta \equiv 0 \mod 2$, then
$$c(w_\alpha \otimes w_\beta) = (-1)^q \left[\bar{\mu}^{\frac{1}{2}} \omega^{2k(2n+1)}\right]^{2(s+n)} R^{2(\alpha-1-n)}(w_\alpha \otimes w_\beta).$$

Thank you!

[AG03] N. Andruskiewitsch and M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003), no. 2, 177–243. MR 1994219 (2004i:16046)

[AG18] N. Andruskiewitsch and J. Giraldi, Nichols algebras that are quantum planes, Linear and Multilinear Algebra 66 (2018), no. 5, 961–991. MR 3775317

[AS98] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3, J. Algebra 209 (1998), no. 2, 658–691. MR 1659895

[CDMM04] C. Călinescu, S. Dăscălescu, A. Masuoka, and C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras, J. Algebra 273 (2004), no. 2, 753–779. MR 2037722

[FGM19] F. Fantino, G. García, and M. Mastnak, On finite-dimensional copointed Hopf algebras over dihedral groups, J. Pure Appl. Algebra 223 (2019), no. 8, 3611–3634. MR 3926230

[GHV11] M. Graña, I. Heckenberger, and L. Vendramin, Nichols algebras of group type with many quadratic relations, Adv. Math. 227 (2011), no. 5, 1956–1989. MR 2803792 (2012f:16077)

[Gn00] M. Graña, On Nichols algebras of low dimension, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 111–134. MR 1800709

[Hec06] I. Heckenberger, The Weyl groupoid of a Nichols algebra of diagonal type, Invent. Math. 164 (2006), no. 1, 175–188. MR 2207786

[Hec09] ______, Classification of arithmetic root systems, Adv. Math. 220 (2009), no. 1, 59–124. MR 2462836

[HLV12] I. Heckenberger, A. Lochmann, and L. Vendramin, Braided racks, Hurwitz actions and Nichols algebras with many cubic relations, Transform. Groups 17 (2012), no. 1, 157–194. MR 2891215

[HV17a] I. Heckenberger and L. Vendramin, A classification of Nichols algebras of semisimple Yetter-Drinfeld modules over non-abelian groups, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 2, 299–356. MR 3605018

[HV17b] ______, The classification of Nichols algebras over groups with finite root system of rank two, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 7, 1977–2017. MR 3656477

[MS00] A. Milinski and H.-J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 215–236. MR 1800714

[NN08] D. Naidu and D. Nikshych, Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups, Comm. Math. Phys. 279 (2008), no. 3, 845–872. MR 2386730

[Shi19] Y.-X. Shi, Finite-dimensional Hopf algebras over the Kac-Paljutkin algebra H_8, Rev. Un. Mat. Argentina 60 (2019), no. 1, 265–298. MR 3981570

[Shi20] ______, Finite dimensional Nichols algebras over Suzuki algebra I: over simple Yetter-Drinfeld modules $A_{N\,2n}^{\mu\lambda}$, arXiv:2011.14274 (2020).

[Shi21] ______, Finite dimensional Nichols algebras over Suzuki algebra II: over simple Yetter-Drinfeld modules of $A_{N\,2n+1}^{\mu\lambda}$, arXiv:2103.06475 (2021).

[Suz98] S. Suzuki, A family of braided cosemisimple Hopf algebras of finite dimension, Tsukuba J. Math. 22 (1998), no. 1, 1–29. MR 1637640