# The Casimir number and the determinant of a fusion category


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- For any simple object $X$ of $\mathcal{C}$, define

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The global dimension of $\mathcal{C}$ is

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Example: $\dim(\mathrm{Rep}(H)) = \mathrm{Tr}_H(S^2)$ for a semisimple Hopf algebra $H$.

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- A fusion category $\mathcal{C}$ over $\Bbbk$ is non-degenerate if $\dim(\mathcal{C}) \neq 0$ in $\Bbbk$.

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- $[c(1)]$ is a positive definite integer matrix, $d_{\mathcal{C}} := \det[c(1)] > 0$ called the determinant of $\mathcal{C}$.

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# Example 2

Let $\mathcal{C}$ be a near-group category over $\Bbbk$. The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ has a basis $G \cup \{X\}$ obey the following multiplication rule:

$$g \cdot h = gh, \; g \cdot X = X \cdot g = X, \; X^2 = \sum_{g \in G} g + \rho X,$$

for all $g, h \in G$ and an integer $\rho \geq 0$.

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- The Casimir number $m_{\mathcal{C}} = \begin{cases} (4|G| + \rho^2)|G|, & 2 \nmid \rho; \\ \frac{1}{2}(4|G| + \rho^2)|G|, & 2 \mid \rho. \end{cases}$

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- $[c(1)] = \begin{bmatrix} M & \mathbf{u} \\ \mathbf{u}^t & \rho^2 + 2|G| \end{bmatrix}$, where $M$ is a square matrix of size $|G|$ whose diagonal elements are all $|G| + 1$ and off-diagonal elements are all 1, $\mathbf{u}$ is a column vector of size $|G|$ whose elements are all $\rho$.

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Let $\mathcal{C}$ be a modular category over a field $\Bbbk$ with isomorphism classes of simple objects $\{X_i\}_{i\in I}$. That is, $\mathcal{C}$ is a spherical fusion category with a braiding $c$ such that the $S$-matrix $S = [s_{ij}]$ is invertible in $\Bbbk$, where $s_{ij} = \mathrm{Tr}(c_{X_j X_i} \circ c_{X_i X_j})$.

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$$h_i : X_j \mapsto \frac{s_{ij}}{\dim(X_i)} \quad \text{for } j \in I$$

defines a homomorphism from $\mathrm{Gr}(\mathcal{C})$ to $\Bbbk$. Note that all eigenvalues of the matrix $[c(1)]$ are $h_i(c(1))$ for $i \in I$. Moreover,

$$h_i(c(1)) = \sum_{j\in I} h_i(X_j)h_i(X_{j^*}) = \sum_{j\in I} \frac{s_{ij}s_{ij^*}}{\dim(X_i)^2} = \frac{\dim(\mathcal{C})}{\dim(X_i)^2}.$$

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It follows that

$$d_{\mathcal{C}} = \prod_{i\in I} h_i(c(1)) = \frac{(\dim\mathcal{C})^{|I|}}{\prod_{i\in I}\dim(X_i)^2}.$$

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  $\Leftrightarrow$ $m_{\mathcal{C}}$ or $d_{\mathcal{C}} \neq 0$ in $\Bbbk$.

# Example 4

Let $\mathcal{C}$ be a Verlinde modular category with isomorphism classes of simple objects $\{X_i\}_{0\leq i\leq n}$. The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ of $\mathcal{C}$ is the truncated Verlinde ring whose multiplication rule is $X_iX_j = \sum_{l=\max\{i+j-n,0\}}^{\min\{i,j\}} X_{i+j-2l}$.

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- $\mathrm{Gr}(\mathcal{C}) \cong \mathbb{Z}[X]/(E_{n+1}(X))$, where

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- The Casimir number $m_{\mathcal{C}} = 2n+4$.
- $\mathcal{C}$ is non-degenerate $\Leftrightarrow$ $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple $\Leftrightarrow$ $2n+4 \neq 0$ in $\Bbbk$ $\Leftrightarrow$ $E_{n+1}(X)$ has no multiple factors in $\Bbbk[X]$.

# Properties of Casimir numbers and determinants

Let $\mathcal{C}$ be a fusion category over $\Bbbk$ with $\mathrm{Irr}(\mathcal{C}) = \{X_i\}_{1\le i\le n}$. Denote $\widetilde{\mathcal{C}}$ the pivotalization of $\mathcal{C}$: $\mathrm{Irr}(\widetilde{\mathcal{C}}) = \{X_i^{\pm}\}_{1\le i\le n}$ and $\mathbf{1}^+ = \mathbf{1}$, $\mathbf{1}^- \otimes \mathbf{1}^- = \mathbf{1}$, $\dim(\mathbf{1}^-) = -1$, and $X_i^{\pm} \otimes \mathbf{1}^- = \mathbf{1}^- \otimes X_i^{\pm} = X_i^{\mp}$.

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- Question: $p \mid d_{\mathcal{C}} \Leftrightarrow p \mid d_{\widetilde{\mathcal{C}}}$ (for any prime $p \neq 2$)?

# The case of semisimple Hopf algebras

## Proposition

*For a semisimple pivotal Hopf algebra $H$, $Rep(H)$ is non-degenerate* $\Leftrightarrow$ $d_{Rep(H)} \neq 0$ *in* $\Bbbk \Leftrightarrow S^2 = id$ *and* $\dim_{\Bbbk}(H) \neq 0$ *in* $\Bbbk$.

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**Proposition**

*Let $H$ be a semisimple and cosemisimple Hopf algebra over $\Bbbk$. If $Gr(H)$ is commutative, then $d_{Rep(H)}$ and $\dim_{\Bbbk} H$ have the same prime factors.*

# The case of semisimple Hopf algebras

**Theorem**

*Let $H$ be a semisimple and cosemisimple Hopf algebra over $\mathbb{k}$ and $D(H)$ the Drinfeld double of $H$. Then $d_{Rep(D(H))}$ and $\dim_{\mathbb{k}} H$ have the same prime factors.*

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Question: For a semisimple Hopf algebra $H$, $p \mid \dim_{\Bbbk} H \Leftrightarrow p \mid d_{\mathrm{Rep}(H)}$?

# Determinants vs. Frobenius-Schur exponents

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- For any prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\xi_N]$, $\dim(\mathcal{C}) \in \mathbb{Z}[\xi_N]/\mathfrak{p}$.

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## Theorem

*Let $\mathcal{C}$ be a spherical fusion over $\mathbb{C}$. Then $N$ and $d_{\mathcal{C}}$ have the same prime factors.*

Remark: For a group $G$ with $|G| \neq 0$ in $\Bbbk$, $|G|$, $\exp(G)$, $d_{\mathrm{Rep}(G)}$ and $m_{\mathrm{Rep}(G)}$ have the same prime factors.

Let $H$ be a f.d. Hopf algebra of finite representation type. Let $r(H)$ be the Green ring of $H$ with a pair of dual pases $\{\delta^*_{[X]}, [X] \mid [X] \in \mathrm{ind}(H)\}$ with respective to the bilinear form $([X],[Y]) = \dim_{\Bbbk} \mathrm{Hom}_H(X, Y^*)$. Consider the Casimir operator

$$c : r(H) \to r(H), c(x) = \sum_{[X]\in \mathrm{ind}(H)} [X]x\delta^*_{[X]}.$$

Then $Imc \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$, its non-negative generator is called the Casimir number of Rep$(H)$ (a generalization of semisimple case).

- $\dim_{\Bbbk}(H)$ divides the casimir number of Rep$(H)$;
- If Hopf algebras $H_1$ and $H_2$ are gauge equivalent, then $r(H_1) \cong r(H_2)$ and lead to the same Casimir numbers.

# Non-semisimple case

## Theorem

*Let $H$ be a Hopf algebra of finite representation type over $\mathbb{k}$. The Green algebra $r(H) \otimes_{\mathbb{Z}} K$ over a field $K$ is Jacobson semisimple iff the Casimir number of $Rep(H)$ is not zero in $K$.*

Remark: If $p \mid dim_{\mathbb{k}}(H)$, then $r(H) \otimes_{\mathbb{Z}} K$ is not Jacobson semisimple for a field $K$ of $charK = p$.

## Theorem

*Let $H$ be a Hopf algebra of finite representation type over $\mathbb{k}$. The Green ring $r(H)$ is Jacobson semisimple iff the Casimir number of $Rep(H)$ is not zero.*

# Non-semisimple case

Let $G$ be a cyclic group of order $p$ and the field $\Bbbk$ has characteristic $p$. Consider the Hopf algebra $\Bbbk G$:

- The Green ring $r(\Bbbk G) \cong \mathbb{Z}[X]/((X-2)E_{p-1}(X))$;
- The Casimir number of $r(\Bbbk G)$ is $2p^2$;
- The Green ring $\mathbb{Z}[X]/((X-2)E_{p-1}(X))$ is semisimple;
- The Green algebra $K[X]/((X-2)E_{p-1}(X))$ is semisimple iff $charK \neq 2, p$;
- If $charK = p$, then the Jacobson radical of $K[X]/((X-2)E_{p-1}(X))$ is generator by $\overline{X^2-4}$;
- If $charK = 2$, then the Jacobson radical of $K[X]/((X-2)E_{p-1}(X))$ is generator by

$$\sum_{i=0}^{[\frac{p-1}{2}]} \binom{p-1-i}{i} (-1)^i \overline{X^{\frac{p+1}{2}-i}}.$$

**Thank you!**