Cohomologies of Lie coalgebras and applications

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(1) Lie coalgebras were defined in [1] and the dual PBW theorem [2] were given by Michaelis.

(2) In the case where both the Lie coalgebra C and its Lie comodule M are finite dimensional, one may construct a cohomology of C with coefficients in M just by dualization in [3].

For the general case we will construct a subcomplex of the Chevalley-Eilenberg cochain complex of the dual Lie algebra of a Lie coalgebra.

W. Michaelis, Lie coalgebras, *Adv. Math.* 38 (1980), 1-54.

W. Michaelis, The dual Poincaré-Birkhoff-Witt theorem, *Adv. Math.* 57 (1985), 93-162.

M. Gerstenhaber and S. D. Shack, Algebras, bialgebras, quantum groups, and algebraic deformations, in "Deformation theory and quantum groups with applications to mathematical physics", *Contemp. Math.* 134 pp. 61-92, Amer. Math. Soc., 1992.

Motivations

(3) Wells [1] conctructed a four-term exact sequence, now called Wells (exact) sequences, to obtain an obstruction class for extensibility of a pair of group automorphisms. Wells maps and Wells sequence for group extensions have been studied further in [2][3]. Extensions of Lie algebra automorphisms has been studied in [4], where sequences of Wells type are also constructed to obtain obstruction classes. We will investigate similar problems in extensions of coderivations and automorphisms of Lie coalgebras.

- C. Wells, Automorphisms of group extensions, *Trans. AMS* 155 (1971) 189-194.
- P. Jin and H.Liu, The Wells exact sequence for the automorphism group of a group extension, *J. Algebra* 324 (2010) 1219-1228.
- I. B. S. Passi, M. Singh and M. K. Yadav, Automorphisms of abelian group extensions, *J. Algebra* 324 (2010) 820-830.
- V. G. Bardakov and M. Singh, Extensions and automorphisms of Lie algebras, *J. Algebra Appl.* 16 (2017) 1750162.

Motivations

(4) Lie bialgebras defined by Drinfel'd [1][2]. How to construct a Lie bialgebra? Lie bialgebra structures on some infinite-dimensional Lie algebras such as the HeisenbergVirasoro algebra, the Witt algebra, Weyl algebras has been described in [3][4]. We will give a way to construct Lie algebras from any Lie coalgebra by the 0-th differential of the cochain complex with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.

- Belavin A. A., Drinfel'd V. G. (1982). Solutions of the classical Yang-Baxter equations for simple Lie algebras. *Funct. Anal. Appl.* 16:159-180.
- Drinfel'd V. G. (1987). Quantum groups, in *Proceedings, International Congress of Mathematicians, August 3-11, 1986, Berkeley, CA* (A.M. Gleason, Ed.), pp. 798-820, Amer. Math. Soc., Providence, RI.
- Michaelis W. (1994). A class of infinite-dimensional Lie bialgebras containing the Virasoro algebra. *Adv. Math.* 107:365-392.
- Liu D., Pei Y., Zhu L. (2012). Lie bialgebra structures on the twisted HeisenbergVirasoro algebra. *J. Algebra* 359:35-48.

The main results of this talk appeared in

1. Du L. and Tan Y. A construction of Lie bialgebras from Lie coalgebras via antisymmetric bilinear forms. *Commu. Algebra* 48(7) (2020): 3170-3183.
2. Du L. and Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.
3. Du L. and Tan Y. Coderivations, abelian extensions and cohomology of Lie coalgebras. *Commu. Algebra* 49(10) (2021): 4519-4542.

Cohomologies of Lie coalgebras

Definition (Michaelis,1980; Majid, 1995)

(1) A **Lie coalgebra** over $\mathbb{F}$ is a pair (C, Δ) of a linear space C and a linear map $\Delta: \ C \to C \otimes C$ subject to the following two conditions.

(i) $\Delta = -\tau \circ \Delta$, and

(ii) $(1 + \xi + \xi^2) \circ (1 \otimes \Delta) \circ \Delta = 0 : \ \ C \to C \otimes C \otimes C$.

The map Δ is called the Lie cobracket of C.

(2) A right **Lie comodule** of a Lie coalgebra (C, Δ) is a pair (M, ρ) of a linear space M and a linear map $\rho: \ M \to M \otimes C$ such that

$$(1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho - (1 \otimes \tau) \circ (\rho \otimes 1) \circ \rho : \ M \to M \otimes C \otimes C.$$

W. Michaelis, *Lie coalgebras*, Adv. Math. 38 (1980), 1-54.

S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.

Cohomologies of Lie coalgebras

In the sequel we shall use the following identification. Let W be a linear space and $n \geq 1$ a positive integer. The antisymmetrization Alt on $\otimes^n W$ is given by

$$\mathrm{Alt}(w_1 \otimes \cdots \otimes w_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \psi_\sigma (w_1 \otimes \cdots \otimes w_n), \; w_i \in W.$$

Then we can identify $\wedge^n W$ with the image $\mathrm{Alt}(\otimes^n W)$ via

$$\mathrm{Alt}(w_1 \otimes \cdots \otimes w_n) = w_1 \wedge \cdots \wedge w_n, \quad w_i \in W.$$

Let $\Delta \in \mathrm{Hom}(W, W \otimes W)$. For brevity we set

$$\Delta^{n,k} \triangleq 1 \otimes \cdots \otimes 1 \otimes \Delta \otimes 1 \otimes \cdots \otimes 1 : \; \otimes^n W \to \otimes^{n+1} W,$$

that is,

$$\begin{aligned} &\Delta^{n,k}(w_1 \otimes \cdots \otimes w_k \otimes \cdots \otimes w_n) \\ =&w_1 \otimes \cdots \otimes w_{k-1} \otimes \Delta(w_k) \otimes w_{k+1} \otimes \cdots \otimes w_n, \; w_i \in W. \end{aligned}$$

Lemma (Du and Tan, 2021)

Let (M,ρ) be a Lie comodule of a Lie coalgebra (C,Δ). Define δ_c^n: $\mathrm{Hom}(M, \wedge^n C) \to \mathrm{Hom}(M, \wedge^{n+1} C)$ by

$$\begin{aligned} \delta_c^0(h) = {} & (h\otimes 1)\circ\rho, \quad h \in \mathrm{Hom}(M, \wedge^0 C) = M^*, \\ \delta_c^n(h) = {} & \frac{1}{2}\sum_{k=1}^{n}(-1)^k \mathrm{Alt}\circ\Delta^{n,k}\circ h + (-1)^{n-1}\mathrm{Alt}\circ(h\otimes 1)\circ\rho, \\ & h\in \mathrm{Hom}(M,\wedge^n C), n\geq 1. \end{aligned} \tag{1}$$

Then the following diagram commutes for each $n \geq 0$,

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}(M, \wedge^n C) \arrow[r, "\delta_c^n"] \arrow[d, hook] & \mathrm{Hom}(M, \wedge^{n+1} C) \arrow[r, "\delta_c^{n+1}"] \arrow[d, hook] & \cdots . \\ \cdots \arrow[r] & \mathrm{Hom}(\wedge^n C^*, M^*) \arrow[r, "\delta^n"] & \mathrm{Hom}(\wedge^{n+1} C^*, M^*) \arrow[r, "\delta^{n+1}"] & \cdots \end{tikzcd}$$

Cohomology of Lie coalgebras

Theorem (Du and Tan, 2021)

Let (M, ρ) *be a Lie comodule of a Lie coalgebra* (C, Δ)*. Then* $(\mathrm{Hom}(M, \wedge^\bullet C), \delta_c^\bullet)$ *is a cochain complex with the differentials given by Lemma (1).*

Proposition (Du and Tan, 2021)

Let (M, ρ) *be a Lie comodule of a Lie coalgebra* (C, Δ)*. Then* $\mathrm{H}_c^1(M, C) = \mathrm{coder}(M, C)/\mathrm{incoder}(M, C)$ *and* $\mathrm{H}_c^2(M, C) \cong \mathcal{E}(M, C)$*, where the space of equivalent classes of extensions of* C *by* M*.*

Du L. and Tan Y. Coderivations, abelian extensions and cohomology of Lie coalgebras. *Commu. Algebra* 49(10) (2021): 4519-4542.

Application 1: Wells maps and Wells sequences

Assume that (C, Δ) is a Lie coalgebra and (M, ρ) is a Lie comodule of (C, Δ). $\mathfrak{g}(C, M)$ is a Lie algebra of all diagonal coderivations of the semi-direct product $C \ltimes M$. We have a linear map Φ: $\mathfrak{g}(C, M) \to \mathrm{End}(\mathrm{H}^2_c(M, C))$ given by

$$(\partial_C, \partial_M) \mapsto \Phi(\partial_C, \partial_M) : \ [h] \mapsto [\Phi(\partial_C, \partial_M)(h)],$$

where $\Phi(\partial_C, \partial_M)(h)$ is given by

$$\Phi(\partial_C, \partial_M)(h) = \ (\partial_C \otimes 1) \circ h + (1 \otimes \partial_C) \circ h - h \circ \partial_M, \ h \in \mathrm{Hom}(M, \wedge^2 C).$$

Theorem (Du and Tan, 2021)

The map Φ is a representation of $\mathfrak{g}(C, M)$ on $\mathrm{H}^2_c(M, C)$.

Du L. and Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 1: Wells maps and Wells sequences

Let $\varepsilon : 0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an extension of C by M. There exists a linear map $\varpi_\varepsilon : \ \mathfrak{g}(C,M) \to \mathrm{H}_c^2(C,M)$ given by

$$(\partial_C, \partial_M) \mapsto \varpi_\varepsilon(\partial_C, \partial_M) \triangleq \Phi(\partial_C, \partial_M)([\mathrm{h}_\varepsilon]),$$

where $(\partial_C, \partial_M) \in \mathfrak{g}(C,M)$.
According to [1], the map ϖ_ε is called **the Wells map** (with respect to coderivations).

Theorem (Du and Tan, 2021)

Let ε*:* $0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ *be an abelian extension of* C *by* M*. Then a pair* $(\partial_C, \partial_M) \in \mathrm{End}(C) \times \mathrm{End}(M)$ *is coder-extensible with respect to* ε *if and only if* $(\partial_C, \partial_M) \in \mathfrak{g}(C,M)$ *and* $\varpi_\varepsilon(\partial_C, \partial_M) = 0$*.*

C. Wells, Automorphisms of group extensions, *Trans. AMS* 155 (1971) 189-194.

Du L. and Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 1: Wells maps and Wells sequences

Fix an abelian extension ε: $0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ of C by M and $t : M \to C$ is a retraction of f. There is a Lie algebra homomorphism $\mathrm{d}_\varepsilon : \mathrm{coder}_C(E) \to \mathrm{coder}(C) \times \mathrm{End}(M)$ given by $\partial_E \mapsto (\overline{\partial_C}, \overline{\partial_M})$ with

$$\overline{\partial_C}(c) \triangleq t(\partial_E(f(c))), \qquad \overline{\partial_M}(m) \triangleq g(\partial_E(e)),$$

where, $c \in C$, $e \in E$ such that $g(e) = m \in M$ and $\mathrm{coder}_C(E) = \{\partial_E \in \mathrm{coder}(E) \mid \partial_E(f(C)) \subseteq f(C)\}$.

Theorem (Du and Tan, 2021)

Assume that $\varepsilon : 0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ is an abelian extension of C by M. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Z}_c^1(M, C) \longrightarrow \mathrm{coder}_C(E) \xrightarrow{\mathrm{d}_\varepsilon} \mathfrak{g}(C, M) \xrightarrow{\varpi_\varepsilon} \mathrm{H}_c^2(M, C),$$

which is called the Wells sequence for ε with respect to coderivations.

Du L., Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 1: Wells maps and Wells sequences

Similarly, for a Lie comodule (M,ρ) is of Lie coalgebra (C,Δ). $\mathfrak{G}(C,M)$ that consists of diagonal automorphisms of $\mathrm{Aut}_{co}(C\ltimes M)$ is a subgroup of $\mathrm{Aut}_{co}(C\ltimes M)$. We have a linear map $\Psi\colon \mathfrak{G}(C,M)\to \mathrm{End}(\mathrm{H}^2_c(C,M))$ given by

$$(u,v)\mapsto \Psi(u,v):\ [\alpha]\mapsto [\Psi(u,v)(\alpha)],$$

where $\Psi(u,v)$ is given by

$$\Psi(u,v)(\alpha)=(u\otimes u)\circ\alpha\circ v^{-1},\ \alpha\in \mathrm{Hom}(M,\wedge^2 C).$$

Theorem (Du and Tan, 2021)

The map Ψ is a representation of the group $\mathfrak{G}(C,M)$ on $\mathrm{H}^2_c(M,C)$.

Du L. and Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 1: Wells maps and Wells sequences

Let $\varepsilon : 0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an extension of C by M. There exists a linear map $\widehat{\varpi}_\varepsilon : \mathfrak{G}(C, M) \to \mathrm{H}^2_c(M, C)$ given by

$$(u, v) \mapsto \widehat{\varpi}_\varepsilon(u, v) \triangleq [\Psi(u, v)(\mathrm{h}_\varepsilon)] - [\mathrm{h}_\varepsilon].$$

The map $\widehat{\varpi}_\varepsilon$ is called **the Wells map** (with respect to automorphisms).

Theorem (Du and Tan, 2021)

Let $\varepsilon : 0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an abelian extension of C by M. Then a pair $(u, v) \in \mathrm{Aut}(C) \times \mathrm{Aut}(M)$ is aut-extensible with respect to ε if and only if $(u, v) \in \mathfrak{G}(C, M)$ and $\widehat{\varpi}_\varepsilon(u, v) = 0$.

Du L. and Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 1: Wells maps and Wells sequences

Fix an abelian extension ε: $0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ of C by M and $t: M \to C$ is a retraction of f. There is a Lie algebra homomorphism a_ε:$\mathrm{Aut}_C(E) \to \mathrm{Aut}_{co}(C) \times \mathrm{Aut}(M)$ with

$$\overline{\gamma_C}(c) = t(\gamma(f(c))), \quad \overline{\gamma_M}(m) = g(\gamma(e)),$$

where, $c \in C$, $e \in E$ such that $g(e) = m \in M$ and $\mathrm{Aut}_C(E) = \{\gamma \in \mathrm{Aut}_{co}(E) \mid \gamma(f(C)) = f(C)\}$.

Theorem (Du and Tan, 2021)

Assume that $\varepsilon : 0 \to C \xrightarrow{f} E \xrightarrow{g} M \to 0$ is an abelian extension of C by M. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Z}_c^1(M, C) \longrightarrow \mathrm{Aut}_C(E) \xrightarrow{\mathrm{a}_\varepsilon} \mathfrak{G}(C, M) \xrightarrow{\widehat{\varpi}_\varepsilon} \mathrm{H}_c^2(M, C).$$

Du L., Tan Y. Wells sequences for abelian extensions of Lie colagebras. *J. Algebra Appl.* 20(8) (2021) Paper No. 2150149, pp 41.

Application 2: A construction of Lie bialgebras from Lie coalgebras

Let $(\mathfrak{g}, \Delta)$ be a Lie coalgebra and $(\mathfrak{g} \otimes \mathfrak{g}, \rho_{\mathrm{ad}})$ is a Lie comodule of $\mathfrak{g}$.
$\delta_c^0 \ : \ (\mathfrak{g} \otimes \mathfrak{g})^* \to \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \ \mathfrak{g})$ is the 0-th differential of the cochain complex with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.
For any $\gamma \in (\mathfrak{g} \otimes \mathfrak{g})^*$ we introduce the following bilinear operation $[\cdot, \cdot]_\gamma$: $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ given by

$$\begin{aligned} [x, y]_\gamma &= \delta_c^0(\gamma)(x \otimes y) = (\gamma \otimes 1)(\rho_{\mathrm{ad}}(x \otimes y)) \\ &= \sum \gamma(x \otimes y_{(1)}) y_{(2)} - \gamma(x_{(2)} \otimes y) x_{(1)}, \quad x, y \in \mathfrak{G}, \end{aligned}$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \Delta(y) = \sum y_{(1)} \otimes y_{(2)}$.

Application 2: A construction of Lie bialgebras from Lie coalgebras

Let $\mathcal{A}^2(\mathfrak{g}) \subseteq (\mathfrak{g} \otimes \mathfrak{g})^*$ be the space of antisymmetric bilinear functions on $\mathfrak{g}$. We have the following

Lemma (Du and Tan, 2020)

Let $(\mathfrak{g}, \Delta)$ be a Lie coalgebra and $\gamma \in \mathcal{A}^2(\mathfrak{g})$. Then for any $x, y \in \mathfrak{g}$ it holds that

$$\begin{aligned}\Delta([x,y]_\gamma) = &\sum \big([x, y_{(1)}]_\gamma \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}]_\gamma \\ &-[y, x_{(1)}]_\gamma \otimes x_{(2)} - x_{(1)} \otimes [y, x_{(2)}]_\gamma\big) \\ =&\mathrm{ad}_x(\Delta(y)) - \mathrm{ad}_y(\Delta(x))\end{aligned}$$

Du L., Tan Y. A construction of Lie bialgebras from Lie coalgebras via antisymmetric bilinear forms. *Commu. Algebra* 48(7) (2020): 3170-3183.

Application 2:A construction of Lie bialgebras from Lie coalgebras

Let $(\mathfrak{g}, \Delta)$ be a Lie coalgebra and $\gamma \in (\mathfrak{g} \otimes \mathfrak{g})^*$. For brevity we introduce $D_\gamma \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^*$ given by

$$\begin{aligned} D_\gamma =&(\gamma \otimes \gamma) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta) \\ =&(\gamma \otimes \gamma) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1 \otimes 1 + 1 \otimes \tau \circ \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta). \end{aligned}$$

Note that the condition $D_\gamma = 0$ has been used in the definition of dual quasitriangular Lie bialgebras (see [1, (8.12)]).

Majid S. (1995). *Foundations of Quantum Group Theory.* Cambridge University Press, Cambridge.

Application 2: A construction of Lie bialgebras from Lie coalgebras

Now we get the following

Theorem (Du and Tan, 2020)

Let $(\mathfrak{g}, \Delta)$ *be a Lie coalgebra and* $\gamma \in \mathcal{A}^2(\mathfrak{g})$. *Then* $(\mathfrak{g}, [\cdot,\cdot]_\gamma)$ *is a Lie algebra if and only if* γ *satisfies*

$$(D_\gamma \otimes 1) \circ (1 \otimes 1 \otimes \Delta) \circ (1 + \xi + \xi^2) = 0 : \quad \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}. \tag{2}$$

In this case, $(\mathfrak{g}, [\cdot,\cdot]_\gamma, \Delta)$ *is a Lie bialgebra.*

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Application 2: A construction of Lie bialgebras from Lie coalgebras

For any Lie coalgebra $(\mathfrak{g}, \Delta)$ we set

$$\mathrm{Lieb}(\mathfrak{g}) := \{\gamma \in \mathcal{A}^2(\mathfrak{g}) \mid (D_\gamma \otimes 1) \circ (1 \otimes 1 \otimes \Delta) \circ (1 + \xi + \xi^2) = 0\},$$
$$\mathrm{Lieb}_0(\mathfrak{g}) := \{\gamma \in \mathcal{A}^2(\mathfrak{g}) \mid D_\gamma = 0\}.$$

The above theorem means that $\gamma \in \mathrm{Lieb}(\mathfrak{g})$ if and only if $(\mathfrak{g}, [\cdot,\cdot]_\gamma, \Delta)$ is a Lie bialgebra.

Application 2: A construction of Lie bialgebras from Lie coalgebras

We consider finite-dimensional Lie coalgebras as following

Corollary (Du and Tan, 2020)

Assume that $(\mathfrak{g}, \Delta)$ *is a finite-dimensional Lie coalgebra and* $\gamma \in \mathrm{Lieb}(\mathfrak{g})$. *Then the dual* $\mathfrak{g}^*$ *of* $(\mathfrak{g}, [\cdot, \cdot]_\gamma, \Delta)$ *is a coboundary Lie bialgebra. Assume further that* $\gamma \in \mathrm{Lieb}_0(\mathfrak{g})$. *Then* $\mathfrak{g}^*$ *is a triangular Lie bialgebra.*

Du L. and Tan Y. A construction of Lie bialgebras from Lie coalgebras via antisymmetric bilinear forms. *Commu. Algebra* 48(7) (2020): 3170-3183.

Application 2: A construction of Lie bialgebras from Lie coalgebras

Any $\gamma \in \mathcal{A}^2(\mathfrak{g}_{3,i}) = \mathfrak{g}_{3,i}^* \wedge \mathfrak{g}_{3,i}^*$ has the form $\gamma = k_1(e_1^* \wedge e_2^*) + k_2(e_1^* \wedge e_3^*) + k_3(e_2^* \wedge e_3^*)$ for some $k_i \in \mathbb{C}$, $i = 1, 2, 3$. We identify γ with the triple $(k_1, k_2, k_3) \in \mathbb{C}^3$ for brevity. We list $\mathrm{Lieb}(\mathfrak{g}_{3,i})$ and $\mathrm{Lieb}_0(\mathfrak{g}_{3,i})$ for each $\mathfrak{g}_{3,i}$ as follows.

Example

(0) $\mathrm{Lieb}_0(\mathfrak{g}_{3,0}) = \mathrm{Lieb}(\mathfrak{g}_{3,0}) = \mathbb{C}^3$.

(1) $\mathrm{Lieb}_0(\mathfrak{g}_{3,1}) = \mathrm{Lieb}(\mathfrak{g}_{3,1}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid k_1 k_3 = 0\}$.

(2) $\mathrm{Lieb}_0(\mathfrak{g}_{3,2}) = \mathrm{Lieb}(\mathfrak{g}_{3,2}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid k_2 = 0\}$.

(3) $\mathrm{Lieb}_0(\mathfrak{g}_{3,3,\alpha}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid (1-\alpha) k_2 k_3 = 0\}$,
$\mathrm{Lieb}(\mathfrak{g}_{3,3,\alpha}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid (1-\alpha^2) k_2 k_3 = 0\}$.

(4) $\mathrm{Lieb}_0(\mathfrak{g}_{3,4}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid k_1 = 0\}$, $\mathrm{Lieb}(\mathfrak{g}_{3,4}) = \mathbb{C}^3$.

(5) $\mathrm{Lieb}_0(\mathfrak{g}_{3,5}) = \{\gamma = (k_1, k_2, k_3) \in \mathbb{C}^3 \mid k_1^2 + k_2^2 + k_3^2 = 0\}$, $\mathrm{Lieb}(\mathfrak{g}_{3,5}) = \mathbb{C}^3$.

Thank You!