Quantum commutative Galois objects and their applications in Brauer groups

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Outline

- Motivation
- The construction of braided Hopf subalgebras
- The exact sequence for Brauer groups
- quantum commutative Galois objects and antoequivalenes

Motivation

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- If $\mathscr{C} \cong \mathscr{D}$, then $Br(\mathscr{C}) \simeq Br(\mathscr{D})$.
- Let (H, R) be a quasitriangular Hopf algebra. Consider ${}_H\mathscr{M}$

Motivation

- An algebra in ${}_H\mathscr{M}$ is called *Azumaya* if

$$
\begin{aligned}
F: A\sharp A^{op} &\longrightarrow End(A), & F(a\sharp\overline{b})(c) &= a(R^2\cdot c)(R^1\cdot b)\\
G: A^{op}\sharp A &\longrightarrow End^{op}(A), & G(\overline{b}\sharp a)(c) &= (R^2\cdot c)(R^1\cdot b)a
\end{aligned}
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  are isomorphic.
- Two Azumaya algebras A and B are Brauer equivalent if there exist finite dimensional H-modules M and N such that

$$A\sharp End(M) \backsimeq B\sharp End(N).$$

  Denote $BR({}_H\mathscr{M})$ by the set of isomorphism classes of Azumaya algebras. Then the Brauer group $Br({}_H\mathscr{M})$ of ${}_H\mathscr{M}$

$$BM(H,R) =: BR({}_H\mathscr{M})/\backsim.$$

- (Zhang, 2004) There exists an exact sequence:

$$1 \longrightarrow Br(k) \longrightarrow BM(H,R) \longrightarrow Gal^{qc}(H^R)$$

Note that we have

$$_H\mathscr{M} \hookrightarrow \mathscr{Z}_l(_H\mathscr{M}) \cong ^H_H \mathscr{Y}\mathscr{D} \cong^{H_R} (_H\mathscr{M}).$$

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- If k is algebraically closed, then

$$BM(H,R) \simeq Aut^{br}({}^H_H\mathscr{Y}\mathscr{D}, {}_H\mathscr{M}) \simeq Gal^{qc}(H_R).$$

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- Proposal: Compute the Brauer group of $B \bowtie H$ by

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- Proposal: Compute the Brauer group of $B \bowtie H$ by

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- *The key point is to characterize $Gal^{qc}((B \bowtie H)_R)$!*

- Main steps:
  1. To find Hopf subalgebra B_Q of $(B \bowtie H)_R$;
  2. To construct $\Omega : Gal^{qc}((B \bowtie H)_R) \longrightarrow Gal^{qc}(B_Q)$;
  3. To compute/ characterize $Gal^{qc}(B_Q)$.

- $E(2)$ is generated by c, x such that

$$c^2 = 1,\ x_i^2 = 0,\ cx_i + x_i c = 0,\ x_1 x_2 + x_2 x_1 = 0, \quad i = 1, 2$$

The quasi-triangular structure

$$\begin{aligned} R &= \frac{1}{2}(1 \otimes 1 + 1 \otimes c + c \otimes 1 - c \otimes c) + \frac{1}{2}(x_1 \otimes cx_2 \\ &\quad + x_1 \otimes x_2 + cx_1 \otimes cx_2 - cx_1 \otimes x_2). \end{aligned}$$

By Majid's transmutation, the braided Hopf algebra $E(2)_R$ with the coalgebra structure

-

$$
\begin{aligned}
&\overline{\Delta}(1) = 1 \otimes 1, \ \overline{\Delta}(cx_1) = 1 \otimes cx_1 + cx_1 \otimes 1, \\
&\overline{\Delta}(cx_2) = 1 \otimes cx_2 + cx_2 \otimes 1, \\
&\overline{\Delta}(x_1x_2) = 1 \otimes x_1x_2 + x_1x_2 \otimes 1 - cx_1 \otimes cx_2 + cx_2 \otimes cx_1, \\
&\overline{\Delta}(c) = c \otimes c + 2x_2 \otimes x_1, \\
&\overline{\Delta}(x_1) = c \otimes x_1 + c - 2x_1 \otimes cx_1x_2, \\
&\overline{\Delta}(x_2) = c \otimes x_2 + c - 2cx_1x_2 \otimes x_1, \\
&\overline{\Delta}(cx_1x_2) = (\mu \otimes \mu)(1 \otimes C \otimes 1)(\overline{\Delta}(c) \otimes \overline{\Delta}(cx_1x_2)).
\end{aligned}
$$

Then $B_Q = k\{1, cx_1, cx_2, x_1x_2\}$ is a braided Hopf subalgebra

$$\overline{S}(1) = 1, \ \overline{S}(cx_1) = -cx_1, \ \overline{S}(cx_2) = -cx_2, \ \overline{S}(x_1x_2) = x_1x_2.$$

However, $k\{1, c\}$ is NOT a braided Hopf subalgebra!!

The construction of braided Hopf subalgebras

- $E(2) \simeq B \bowtie k[Z_2]$. Then

$$B_Q \hookrightarrow E(2)_R \simeq (B \bowtie k[Z_2])_R$$

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- Problem 1: Consider Radford's biproduct $B \bowtie H$, is there such a braid Hopf subalgebra that comes from B?
- Problem 2: What is the role of this braid Hopf subalgebra in the computation of Brauer groups?

- (Wang, 1999) Let $(B \bowtie H, R)$ be quasi-triangular. Then (H, r) is quasi-triangular, where $r = (p \otimes 1 \otimes p \otimes 1)(R)$; B has a quasi-triangular structure Q in $({}_H\mathscr{M}, r)$, where

$$Q = (1 \otimes \varepsilon \otimes 1 \otimes \varepsilon)(R) \in B \otimes B.$$

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- Since ${}_B({}_H\mathscr{M}) \cong {}_{B\bowtie H}\mathscr{M}$, then as braided tensor categories

$$({}_B({}_H\mathscr{M}), Q) \cong ({}_{B\bowtie H}\mathscr{M}, R).$$

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- Question 1: Can (B, Q) induce a Hopf algebra B_Q in ${}_B({}_H\mathscr{M})$?

The construction of braided Hopf subalgebras

- Let (H, R) be triangular.

Definition 1

A braided Hopf algebra B in ${}_H\mathscr{M}$ is quasi-triangular if there exists an invertible element $Q := Q^1 \otimes Q^2$ in $B \otimes B$ such that

$$(\Delta \otimes 1)Q = r^1 \cdot Q^1 \otimes T^1 \otimes Q^2(r^2 \cdot T^2)$$
$$(1 \otimes \Delta)Q = Q^1 T^1 \otimes T^2 \otimes Q^2$$
$$r^1 \cdot (Q^1 b_{(1)}) \otimes Q^2(r^2 \cdot b_{(2)}) = (r^1 \cdot b_{(2)})Q^1 \otimes b_{(1)}(r^2 \cdot Q^2)$$
$$h_2 \cdot Q^1 \otimes h_1 \cdot Q^2 = Q^1 \otimes Q^2 \varepsilon(h),$$

where $Q = T$, etc.

- $B \bowtie H$ has the quasi-triangular structure $\mathbf{R}$, where

$$\mathbf{R} := r^1 \cdot Q^1 \otimes t^1 \otimes Q^2 \otimes r^2 t^2.$$

- The braided version of Majid's Transmutation.

Proposition 2

Let (B, Q) be a quasi-triangular braided Hopf algebra. Then there is a Hopf algebra B_Q in ${}_B({}_H\mathscr{M})$, where $B_Q = B$ as a linear space and as an object in ${}_B({}_H\mathscr{M})$ by the braided adjoint action

$$a \blacktriangleright b = a_{(1)}(r^2 \cdot b)S(r^1 \cdot a_{(2)}), \text{ for all } a, b \in B.$$

The algebra structure and counit in B_Q coincide with those of B. The comultiplication and antipode are as follow

$$\overline{\Delta}(b) = b_{(1)}S(Q^2) \otimes Q^1 \blacktriangleright b_{(2)}, \quad \overline{S}(b) = Q^2 S(Q^1 \blacktriangleright b),$$

where $\Delta(b) = b_{(1)} \otimes b_{(2)}$ in B.

- In the case of $E(2)$, $B_Q = k\{1, cx_1, cx_2, x_1x_2\}$

$$\overline{\Delta}(1) = 1 \otimes 1,\ \overline{\Delta}(cx_1) = 1 \otimes cx_1 + cx_1 \otimes 1,$$
$$\overline{\Delta}(cx_2) = 1 \otimes cx_2 + cx_2 \otimes 1,$$
$$\overline{\Delta}(x_1x_2) = 1 \otimes x_1x_2 + x_1x_2 \otimes 1 - cx_1 \otimes cx_2 + cx_2 \otimes cx_1.$$
$$\overline{S}(1) = 1,\ \overline{S}(cx_1) = -cx_1,\ \overline{S}(cx_2) = -cx_2,\ \overline{S}(x_1x_2) = x_1x_2.$$

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  Where

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- B_Q is commutative and cocommutative, and isomorphic to the braided tensor product of $B_1 = k\{1, cx_1\}$ and $B_2 = k\{1, cx_2\}$.

- In general, by braided Majid's Transmutation, we also have the Hopf algebra B_Q in ${}_B({}_H\mathscr{M})$ and so in ${}_{B\bowtie H}\mathscr{M}$.

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The construction of braided Hopf subalgebras

- In general, by braided Majid's Transmutation, we also have the Hopf algebra B_Q in ${}_B({}_H\mathscr{M})$ and so in ${}_{B\bowtie H}\mathscr{M}$.
- Other examples, and the notion of quasi-triangular ... in a braided tensor category.
- Remarks on Majid's quasi-triangular braided Hopf algebras.

The exact sequence for Brauer groups

Definition 3

Let A be a right B_Q-comodule algebra. A is called a right braided B_Q-Galois object if A is faithfully flat and the morphism

$$\beta : A \otimes A \longrightarrow A \otimes B_Q, \; a \otimes b \longmapsto ab_{[0]} \otimes b_{[1]}$$

is an isomorphism.

- By Schauenburg's result, a bi-Galois object A induces autoequivalence:

$$A \square - : {}^{B_Q}({}_B({}_H\mathscr{M})) \longrightarrow {}^{B_Q}({}_B({}_H\mathscr{M}))$$

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- Question 2:

$$Gal^{qc}((B \bowtie H)_R \longrightarrow \ Gal^{qc}(B_Q)?$$

The exact sequence for Brauer groups

- We have

Definition 4

A braided bi-Galois object A is called quantum commutative if

$$\begin{aligned} ab &= (a_{(-1)} \blacktriangleright (r^2 \cdot b))(r^1 \cdot a_{(0)}) \\ a_{[0]} \otimes a_{[1]} &= Q^2 \blacktriangleright [p^2 \cdot (T^1 \blacktriangleright a_{[0]})] \otimes p^1 \cdot (Q^1 a_{[-1]} T^2), \end{aligned}$$

for all $a, b \in A$.

Lemma 5

Let (B, Q) be a quasitriangular braided Hopf algebra in ${}_H\mathscr{M}$. Then there is a group homomorphism

$$\Omega : Gal^{qc}((B \bowtie H)_{\mathbf{R}}) \longrightarrow Gal^{qc}(B_Q), \quad A \longmapsto A'.$$

The exact sequence for Brauer group

- By Zhang's exact sequence,

$$Br({}_B({}_H\mathscr{M}), Q) \simeq Br({}_{B\bowtie H}\mathscr{M}, \mathbf{R}) \longrightarrow Gal^{qc}((B \bowtie H)_{\mathbf{R}}).$$

Thus we obtain the following proposition:

Proposition 6

Let (B, Q) be a quasitriangular braided Hopf algebra. Then there is a group homomorphism

$$\pi : Br({}_B({}_H\mathscr{M}), Q) \longrightarrow Gal^{qc}(B_Q).$$

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The exact sequence for Brauer groups

- About $Ker\pi$

Proposition 7

Let (H, R) be triangular, and (B, Q) a quasitriangular braided Hopf algebra. Then there are the following group homomorphisms

$$Br({}_H\mathscr{M}, R) \hookrightarrow Br({}_B({}_H\mathscr{M}), Q) \longrightarrow Gal^{qc}(B_Q).$$

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Proposition 7

Let (H, R) be triangular, and (B, Q) a quasitriangular braided Hopf algebra. Then there are the following group homomorphisms

$$Br({}_H\mathscr{M}, R) \hookrightarrow Br({}_B({}_H\mathscr{M}), Q) \longrightarrow Gal^{qc}(B_Q).$$

- This is the result of Cuadra and Femić , if B is comm. and cocommu, and $Q = 1 \otimes 1$. However, their method can not be valid in the case $(E(2), R)$ since

$$Q = 1 \otimes 1 + cx_1 \otimes cx_2.$$

- $$^{B_Q}\left({}_B({}_H\mathscr{M})\right) \hookrightarrow {}^{(B\bowtie H)_R}\left({}_B({}_H\mathscr{M})\right) \cong {}^{(B\bowtie H)_R}\left({}_{B\bowtie H}\mathscr{M}\right) \cong {}^{B\bowtie H}_{B\bowtie H}\mathscr{YD}$$

- $$ {}^{B_Q}({}_B({}_H\mathscr{M})) \hookrightarrow {}^{(B\bowtie H)_R}({}_B({}_H\mathscr{M})) \cong {}^{(B\bowtie H)_R}({}_{B\bowtie H}\mathscr{M}) \cong {}^{B\bowtie H}_{B\bowtie H}\mathscr{Y}\mathscr{D} $$
- We have

Definition 8

Let B be a braided Hopf algebra in ${}_H\mathscr{M}$. A left B-module $(M, \cdot, \blacktriangleright)$ in ${}_H\mathscr{M}$ is called a categorical Yetter-Drinfeld module if (M, ρ^L) is a left B-comodule satisfying the following condition:

$$\begin{aligned} & a_{(1)}(r^2 \cdot m_{(-1)}) \otimes (r^1 \cdot a_{(2)}) \blacktriangleright m_{(0)} \\ = \; & (a_{(1)} \blacktriangleright (r^2 \cdot m))_{(-1)}(p^2 r^1 \cdot a_{(2)}) \otimes p^1 \cdot (a_{(1)} \blacktriangleright (r^2 \cdot m)_{(0)}), \end{aligned}$$

for all $a \in B$ and $m \in M$, where $\rho^L(m) = m_{(-1)} \otimes m_{(0)}$.

- Bespalov's result:

Lemma 9

(Bespalov, 97) Let B be a braided Hopf algebra with bijective antipode in ${}_H\mathscr{M}$. Then ${}^B_B\mathscr{Y}\mathscr{D}({}_H\mathscr{M})$ has a braiding $C_{V,W}$:

$$C_{V,W}(v \otimes w) = v_{(-1)} \blacktriangleright (r^2 \cdot w) \otimes r^1 \cdot v_{(0)},$$

where $v \in V$ and $w \in W$.

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where $v \in V$ and $w \in W$.

- The above quantum-commutative compatible conditions is just braided-commutative.

- We prove that

Theorem 10

Let (B,Q) be a quasi-triangular braided Hopf algebra with bijective antipode. Then there exist the following braided tensor equivalences

$$\mathscr{Z}_L({}_B({}_H\mathscr{M})) \cong {}^B_B\mathscr{YD}({}_H\mathscr{M}) \cong {}^{B_Q}({}_B({}_H\mathscr{M})) \cong {}^{B_Q}({}_{B\bowtie H}\mathscr{M}),$$

where $\mathscr{Z}_L({}_B({}_H\mathscr{M}))$ is the full subcategory of the Drinfeld center $\mathscr{Z}_l({}_B({}_H\mathscr{M}))$ with the following condition on the half-braiding $\nu_{U,-}$: for any object V in ${}_H\mathscr{M}$ with trivial action (through counit) , $\nu_{U,V}$ coincides with braiding $C_{U,V}$ in ${}_H\mathscr{M}$.

- As a braided tensor cateogry

$$({}_B({}_H\mathscr{M}), Q) \hookrightarrow ({}^B_B\mathscr{Y}\mathscr{D}({}_H\mathscr{M}), C).$$

Denote by $Aut^{br}({}^B_B\mathscr{Y}\mathscr{D}({}_H\mathscr{M}), {}_B({}_H\mathscr{M}))$ the group of isomorphism classes of braided autoequivalences of ${}^B_B\mathscr{Y}\mathscr{D}({}_H\mathscr{M})$ trivializable on ${}_B({}_H\mathscr{M})$.

Theorem 11

Let (B, Q) be a f. d. quasitriangular braided Hopf algebra. Then

$$\Psi : Gal^{qc}(B_Q) \simeq Aut^{br}({}^B_B\mathscr{Y}\mathscr{D}({}_H\mathscr{M}), {}_B({}_H\mathscr{M})), \quad A \longmapsto A\Box -$$

Thank you!