Higher Degree Forms and Tensors

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Based on joint works with 黄华林，吕华军，张驰

- Higher degree form, symmetric tensor
- Classification and canonical forms
- Harrison center
- LDS/Central/Diagonalizable forms
- An algorithm for direct sum decomposition
- Further questions
- References

Tensor : Hamilton (1846), W. Voigt (1896): current usage

Definitions of tensor:

- multidimensional arrays: matrix of higher degree

  $a_{i_1 i_2 \cdots i_d}$: $1 \le i_1, \cdots, i_d \le n$ $\qquad A = (a_{i_1 \cdots i_d})_{n \times \cdots \times n}$

- multilinear map:

  $T: V \times V \times \cdots \times V \longrightarrow \mathbb{K}$.

- elements in $V^* \otimes \cdots \otimes V^*$

- certain geometric objects independent of the choice of coordinates.

4 Higher degree forms

$\mathbb{k}$: field. $V/\mathbb{k}$. $\dim V = n$ $\quad \mathrm{char}\,\mathbb{k} = 0$ or $> d$

$V = \mathbb{k}^n$. $\quad \{e_i\}_{i=1}^n$ basis.

$A = (a_{i_1 i_2 \cdots i_d})_{n\times n \cdots \times n}$ symmetric tensor

$\theta : V \times V \times \cdots \times V \longrightarrow \mathbb{k}$.

$\theta(e_{i_1}, \ldots, e_{i_d}) \triangleq a_{i_1 i_2 \cdots i_d}$

symmetric d-linear form.

$f: V \to k$

$f(v) = \theta(v, v, \ldots, v)$

$f(x_1, \ldots, x_n) = \theta\left(\sum_{i=1}^n x_i e_i, \ldots, \sum_{i=1}^n x_i e_i\right)$

variables is a homogeneous polynomial of degree d.

5 Homogeneous Polynomials

$$f(x_1, \ldots, x_n) = \sum_{\lambda_1+\cdots+\lambda_n=d} b_{\lambda_1\cdots\lambda_n} x_1^{\lambda_1}\cdots x_n^{\lambda_n}$$

$$b_{\lambda_1\cdots\lambda_n} = \frac{d!}{\lambda_1!\cdots\lambda_n!} a_{i_1\cdots i_d}$$

$$(i_1, \ldots, i_d) \in \Lambda(\lambda_1, \ldots, \lambda_n) \overset{\text{``}}{=} \#\{s \mid i_s = u\} = \lambda_u$$

symmetric multilinear form $\Longleftrightarrow$ higher degree matrices

$\shortparallel$

symmetric tensor $\Updownarrow$ $\Updownarrow$

homogeneous polynomials = higher degree forms

$V_{n,d}$: n = dim V = # of variables

d = degree

- Classification of higher degree forms under changing basis?
- Normal form?

$\Longleftrightarrow$ determine all orbits : "congruently" $GL_n(\mathbb{k}) \curvearrowright \left\{ \begin{array}{l} \text{higher linear} \\ \text{forms} \end{array} \right\}$

· Algebraic geometry:
  classification of hypersurfaces of degree d ...

· Invariant theory

· representation theory

<u>Harrison center</u> : Harrison (1975)

$$Z(\theta) = \{ \varphi \in \mathrm{End}\, V \mid \theta(\varphi v_1, v_2, \cdots, v_d) = \theta(v_1, \varphi v_2, \cdots, v_d) \quad \forall v_1, \cdots, v_d \in V \}$$

$$Z(f) = \{ X \in \mathbb{K}^{n\times n} \mid (H_f X)^T = H_f X \}$$

H_f: Hessian matrix: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$

$$Z(A) = \{ X \in \mathbb{K}^{n\times n} \mid X^T A^{(i_3 \cdots i_d)} = A^{(i_3 \cdots i_d)} X, \ \forall 1 \le i_3, \cdots, i_d \le n \}$$

<u>Rmk</u>: generalisation of symmetric matrixes

$d=2$, $A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{n\times n}$ $Z(A) = \{\text{symmetric matrices}\}$

8 Direct sum of higher degree form

decomposable
$\parallel$
(Sebastiani-Thom type)

$$f(x_1, \cdots, x_n) = g(y_1, \cdots, y_r) + h(y_{r+1}, \cdots, y_n)$$

$$\Longleftrightarrow \quad V = V_1 \oplus V_2 , \qquad \theta = \theta|_{V_1} + \theta|_{V_2}$$

orthogonal decomposition.

$$\theta(v_1, v_2, * \cdots *) = 0 . \qquad \forall\, v_1 \in V_1,\ v_2 \in V_2$$

$$\Longleftrightarrow \quad A = (a_{i_1 \cdots i_d})_{n \times n \times \cdots \times n}$$ congruent to block diagonal.

f is <u>diagonalizable</u> if $f = \lambda_1 y_1^d + \cdots + \lambda_n y_n^d$.

<u>Proposition</u> (Harrison 1975) f nondegerate

(1) $Z(f)$ commutative

(2) $f = f_1 + f_2 \Longrightarrow Z(f) = Z(f_1) \times Z(f_2)$

(3) $\{$direct sum decomp. of $f\} \xleftrightarrow{1-1} \{$orthogonal idempotent decomp. of $1_{Z(f)}\}$.

(4) f indecomp. $\Longleftrightarrow$ $Z(f)$ local

(5) f is diagonalizable $\Longleftrightarrow$ $Z(f) \simeq \mathbb{K} \times \cdots \times \mathbb{K}$

(6) decomp. of f is unique

(7) $K / \mathbb{K}$. $\quad Z(f_K) \simeq Z(f) \otimes_{\mathbb{K}} K$

THM f nondegenerate. $Z(f)$ is non semi-simple $\Longleftrightarrow$ f is LDS.

LDS form: (limit of direct sums)

$$f(x_1,\cdots,x_n)=\sum_{i=1}^{l} x_i \frac{\partial h(x_{l+1},\cdots,x_{2l})}{\partial x_{l+i}}+g(x_{l+1},\cdots,x_n)$$

$$\left(=\lim_{t\to 0}\frac{1}{t}\Big[h(\underbrace{tx_1+x_{l+1}}_{y_1},\cdots,\underbrace{tx_l+x_{2l}}_{y_l})-h(x_{l+1},\cdots,x_{2l})+t\,g(\underbrace{x_{l+1}}_{y_{l+1}},\cdots,\underbrace{x_n}_{y_n})\Big]\right)$$

$t\neq 0$
direct sum

W. Buczy´nska, J. Buczy´nski, J. Kleppe and Z. Teitler, Apolarity and direct sum decomposability of polynomials. Michigan Math. J. 64 (2015) 675-719.

11 Central form

<u>Thm</u> The set of central forms is an open dense subspace of the space of higher degree forms.

<u>central form</u> : $Z(f) = \mathbb{K}$

- eg. $f = x_1 x_2^{d-1} + \cdots + x_{n-1} x_n^{d-1} + x_n x_1^{d-1}$ is central form.

- 不可约型的例子很难给出，利用中心很容易给出例子.

12 Characterization of central form

$J(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$, Jacobian ideal

<u>THM</u> $f \in V_{n,d}$ nondegenerate. TFAE:

(1) f can be reconstructed from $J(f)$

(2) f is a central form

(3) $\mathrm{rk}\, B = n^2 - 1$.

(4) f is indec. non LDS form.

- M. Fedorchuk, Direct sum decomposability of polynomials and factorization of associated forms. Proc. London Math. Soc. (3) 120 (2020) 305-327.
- Z. Wang, On homogeneous polynomials determined by their Jacobian ideal. Manuscripta Math. 146 (2015) 559-574.

Prop $\mathrm{Diag}_{n,d}$ is a proper closed subset of $U_{n,d}$.

THM $f \in V_{n,d}$ nondegenerate. TFAE:

(1) $f \in \mathrm{Diag}_{n,d}$

(2) $Z(f) \simeq \mathbb{K}^n$

(3) f smooth & $\dim Z(f) = n$

(4) $Z(f)$ s.simple, $\dim Z(f) = n$

(5) $\dim Z(f) = n$. $Z(f)$ consists of diagonizable matrices

(6) $\dim Z(f) = n$. $Z(f)$ has a basis consisting of matrices of rank 1 and trace 1.

14 An algorithm for direct sum decomposition

Algorithm: *Take an arbitrary* $f \in \mathrm{V}_{n,d}$. *Denote the associated symmetric tensor by* A.

Step 1. *Compute* $\mathrm{Rank}\{A_1, \cdots, A_n\}$. *If it is* n*, then* f *is nondegenerate and continue; otherwise, take a linearly independent set of maximal size, reduce variables and make* f *nondegenerate in lower dimension situation, then continue.*

Step 2. *Solve the associated linear equations and get a basis* $(P_i)_{1\leq i\leq \dim Z(f)}$ *of the center* $Z(f)$.

Step 3. *Upper-triangularize* $(P_i)_{1\leq i\leq \dim Z(f)}$ *simultaneously, and get a set of uppertriangular matrices* $(Q_i)_{1\leq i\leq \dim Z(f)}$. *Let* $Z' = \oplus_{1\leq i\leq \dim Z(f)} \Bbbk Q_i$ *denote the conjugate algebra of* $Z(f)$.

Step 4. *Take the diagonal* α_i *of each* Q_i. *By the well known theorem of Jordan decomposition,* α_i *is a polynomial of* Q_i *and so an element of* Z'. *Determine a complete set of primitive orthogonal idempotents of* Z' *which are linearly spanned by the* α_i*'s. Write each* α_i *as a row vector and put them into a matrix* C. *Then a set of primitive orthogonal idempotents are obtained by a row echelon reduction of* C. *By the reverse conjugation of Step 3, get a complete set of primitive orthogonal idempotents, denoted by* $(\epsilon_j)_{1\leq j\leq \dim Z(f)}$*, of* $Z(f)$.

Step 5. *Decompose the form* f *according to the complete set* $(\epsilon_j)_{1\leq j\leq \dim Z(f)}$ *of primitive orthogonal idempotents.*

15 An example: the Keet-Saxena cubic forms

eg. $f = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \le i < j \le n} 2 a_{ij} x_i x_j \in \mathbb{k}[x_i, a_{ij}]$

- $\dim Z(f) = 1 + \frac{n(n+1)(n+2)}{6}$ · $Z(f)$ local, radical square 0
- f is indecomposable LDS form.

Rmk. · Conjecture (O'Ryan-Shapiro '03): $\dim Z(f) < n$. $\forall f \in V_{n,d}$

· Conjecture (Saxena '05): $\dim Z(f) \le (d-1)n$

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- M. O' Ryan, D.B. Shapiro, Centers of higher degree forms. Linear Alg. Appl. 371 (2003) 301-314.

16 References

1. 黄华林，吕华军，叶郁，张驰：Diagonalizable higher degree forms and symmetric tensors. Linear Alg. Appl. 613 (2021) 151-169.

2. 黄华林，吕华军，叶郁，张驰：Centers and direct sum decompositions of higher degree forms. **arXiv:2009.12014.**

17 Further problems

- Multilinear forms and congruence of tensors
- Nilpotency of centers and singularity
- Waring's problem of higher degree forms
- Composition of higher degree forms
- Multilinear algebra and hypermatrices
- Rank, determinant, eigenvalue … for matrices of higher degree
- ……

谢 谢！