On Centers of Braided Tensor Categories[1]

Zhimin Liu

Fudan University

Chongqing, August, 2021

[1] This is a joint work with Shenglin Zhu

Tensor categories

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

   satisfying certain axioms.

$$\begin{tikzcd}[column sep=small]
& ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "a_{W,X,Y}\otimes id_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W\otimes (X\otimes Y))\otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & & (W\otimes X)\otimes (Y\otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"'] \\
W\otimes ((X\otimes Y)\otimes Z) \arrow[rr, "id_W\otimes a_{X,Y,Z}"] & & W\otimes (X\otimes (Y\otimes Z))
\end{tikzcd}, \quad
\begin{tikzcd}
& X\otimes Y & \\
(X\otimes \underline{1})\otimes Y \arrow[ur, "r_X\otimes id_Y"] \arrow[rr, "a_{X,1,Y}"] & & X\otimes (\underline{1}\otimes Y) \arrow[ul, "id_X\otimes l_Y"']
\end{tikzcd},$$

W, X, Y, Z in $\mathcal{C}$.

MacLane's strictness theorem: Any monoidal category is equivalent to a strict one. we assume that the monoidal categories considered are all strict.

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
   Evaluation $ev_X : X^* \otimes X \to 1$, coevaluation $coev_X : 1 \to X \otimes X^*$,

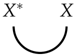
   The maps below are identities:
   $X = 1 \otimes X \to X \otimes X^* \otimes X \to X \otimes 1 = X$,
   $X^* = X^* \otimes 1 \to X^* \otimes X \otimes X^* \to 1 \otimes X^* = X^*$.

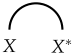
Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.

   Evaluation $ev_X : X^* \otimes X \to 1$, coevaluation $coev_X : 1 \to X \otimes X^*$,

   denoted by $ev_X =$ 

, $coev_X =$ 

,

   The maps below are identities:

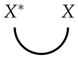
   $X = 1 \otimes X \to X \otimes X^* \otimes X \to X \otimes 1 = X$,

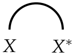
   $X^* = X^* \otimes 1 \to X^* \otimes X \otimes X^* \to 1 \otimes X^* = X^*$.

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

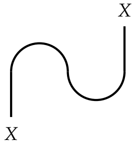
   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
   Evaluation $ev_X : X^* \otimes X \to 1$, coevaluation $coev_X : 1 \to X \otimes X^*$,

   denoted by $ev_X =$ 

, $coev_X =$ 

,

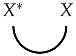
   The maps below are identities:
   $X = 1 \otimes X \to X \otimes X^* \otimes X \to X \otimes 1 = X,$
   $X^* = X^* \otimes 1 \to X^* \otimes X \otimes X^* \to 1 \otimes X^* = X^*.$

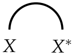
   i.e. 

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

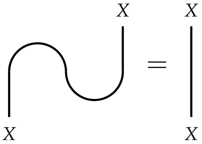
   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
   Evaluation $ev_X : X^* \otimes X \to 1$, coevaluation $coev_X : 1 \to X \otimes X^*$,

   denoted by $ev_X =$ 

, $coev_X =$ 

,

   The maps below are identities:
   $X = 1 \otimes X \to X \otimes X^* \otimes X \to X \otimes 1 = X$,
   $X^* = X^* \otimes 1 \to X^* \otimes X \otimes X^* \to 1 \otimes X^* = X^*$.

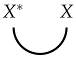
   i.e. 

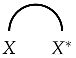
,

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

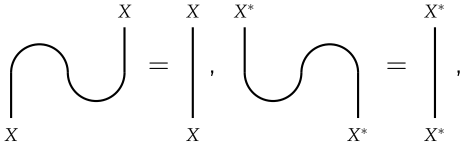
   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
   Evaluation $ev_X : X^* \otimes X \to 1$, coevaluation $coev_X : 1 \to X \otimes X^*$,

   denoted by $ev_X =$ 

   , $coev_X =$ 

   ,

   The maps below are identities:
   $X = 1 \otimes X \to X \otimes X^* \otimes X \to X \otimes 1 = X$,
   $X^* = X^* \otimes 1 \to X^* \otimes X \otimes X^* \to 1 \otimes X^* = X^*$.

   i.e. 

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
3. k-linear, $- \otimes -$ of morphism is bilinear,
   abelian: we can talk about kernels and cokernels,
   locally finite: every object has finite length, $\mathrm{Hom}\,(X, Y)$ is f.d.

Tensor categories

1. Monoidal structure $(\mathcal{C}, \otimes, a, 1, l, r)$
   - $\mathcal{C}$ –category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ –bifunctor, and unit object $1 \in \mathcal{C}$,
   - associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
   - unit constraints $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$,

   satisfying certain axioms.
2. Rigid: every object X has duals $X^*, {}^*X$.
3. k-linear, $- \otimes -$ of morphism is bilinear,
   abelian: we can talk about kernels and cokernels,
   locally finite: every object has finite length, $\mathrm{Hom}(X, Y)$ is f.d.

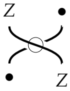
Definition

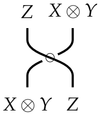
A category $\mathcal{C}$ with such structures and properties is called a multi-tensor category. If in addition $\mathrm{End}(1) \cong k$, then $\mathcal{C}$ is called a tensor category.

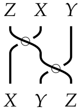
Drinfeld Center (Drinfeld; Joyal, Street)

A construction of braided monoidal category $\mathcal{Z}_l(\mathcal{C})$ starting with any monoidal category $\mathcal{C}$.

- Objects of $\mathcal{Z}_l(\mathcal{C})$: $(Z, \gamma_{Z,\bullet})$ consisting of an object $Z \in \mathcal{C}$ and a natural isomorphism

$\gamma_{Z,\bullet} =$ 

, such that 

$=$ 

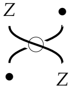
, for all $X, Y \in \mathcal{C}$.

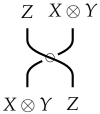
- Tensor product: $(Z, \gamma_{Z,\bullet}) \otimes (Z', \gamma_{Z',\bullet}) = (Z \otimes Z', (\gamma_{Z,\bullet} \otimes 1)(1 \otimes \gamma_{Z',\bullet}))$,
- Braiding: $\gamma_{Z,Z'} : Z \otimes Z' \xrightarrow{\cong} Z' \otimes Z$.

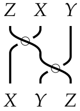
Drinfeld Center (Drinfeld; Joyal, Street)

A construction of braided monoidal category $\mathcal{Z}_l(\mathcal{C})$ starting with any monoidal category $\mathcal{C}$.

- Objects of $\mathcal{Z}_l(\mathcal{C})$: $(Z, \gamma_{Z,\bullet})$ consisting of an object $Z \in \mathcal{C}$ and a natural isomorphism

$\gamma_{Z,\bullet} =$ 

, such that 

$=$ 

, for all $X, Y \in \mathcal{C}$.

- Tensor product: $(Z, \gamma_{Z,\bullet}) \otimes (Z', \gamma_{Z',\bullet}) = (Z \otimes Z', (\gamma_{Z,\bullet} \otimes 1)(1 \otimes \gamma_{Z',\bullet}))$,
- Braiding: $\gamma_{Z,Z'} : Z \otimes Z' \xrightarrow{\cong} Z' \otimes Z$.

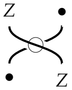
Remarks

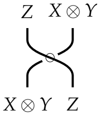
1. *(Muger,Etingof,Nikshych,Ostrik) Assume that $\mathcal{C}$ is multi-fusion, then $\mathcal{Z}_l(\mathcal{C})$ is also multi-fusion.*
2. *(Etingof,Ostrik) Assume that $\mathcal{C}$ is a finte tensor catgory, then $\mathcal{Z}_l(\mathcal{C})$ is also a finte tensor catgory.*

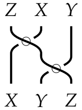
Drinfeld Center (Drinfeld; Joyal, Street)

A construction of braided monoidal category $\mathcal{Z}_l(\mathcal{C})$ starting with any monoidal category $\mathcal{C}$.

- Objects of $\mathcal{Z}_l(\mathcal{C})$: $(Z, \gamma_{Z,\bullet})$ consisting of an object $Z \in \mathcal{C}$ and a natural isomorphism

$\gamma_{Z,\bullet} =$ 

, such that 

$=$ 

, for all $X, Y \in \mathcal{C}$.

- Tensor product: $(Z, \gamma_{Z,\bullet}) \otimes (Z', \gamma_{Z',\bullet}) = (Z \otimes Z', (\gamma_{Z,\bullet} \otimes 1)(1 \otimes \gamma_{Z',\bullet}))$,
- Braiding: $\gamma_{Z,Z'} : Z \otimes Z' \xrightarrow{\cong} Z' \otimes Z$.

Examples

- *Let H be a Hopf algebra. Then $\mathcal{Z}_l({}_H\mathcal{M}) = {}^H_H\mathcal{YD}$.*
- *$H = kG$, G – finite group, then*

$$\mathrm{Irr}(\mathcal{Z}_l({}_{kG}\mathcal{M})) = \{(x, \rho) \mid x \in \Gamma, \rho \in \mathrm{Irr}(C_G(x))\},$$

where Γ is the set of representatives of conjugacy classes.

Motivations

Question

For a Hopf algebra H, classify the irreducible objects of ${}^H_H\mathcal{YD}$?

Motivations

Question

For a Hopf algebra H, classify the irreducible objects of ${}^H_H\mathcal{YD}$?

For general Hopf algebras H, this problem seems to be very difficult.

Motivations

Question

For a Hopf algebra H, classify the irreducible objects of ${}^H_H\mathcal{YD}$?

For general Hopf algebras H, this problem seems to be very difficult.
But for quasi-triangular (weak) Hopf algebras (H, R),...

Motivations

Question

For a Hopf algebra H, classify the irreducible objects of ${}^{H}_{H}\mathcal{YD}$?

For general Hopf algebras H, this problem seems to be very difficult.
But for quasi-triangular (weak) Hopf algebras (H, R),...

1. Dijkgraaf, Pasquier, and Roche(1991): Present the structure theory of kG-YD-modules;
2. Reshetikhin and Semenov-Tian-Shansky(1988): Illustrates that $D(H) \cong (H \otimes H)^{\sigma}$ for a factorizable Hopf algebra H;
3. L. and Zhu(2019): Present the structure of YD-modules over semisimple and cosemisimple QT Hopf algebra.
4. For a QT weak Hopf algebras H,
   Liu G. and Zhu H.(2012): constructed a braided Hopf algebra B in $\mathcal{C} = {}_{H}\mathcal{M}$;
   Zhu H. and Zhang Y.(2015): prove that ${}^{H}_{H}\mathcal{YD} \cong B\text{-Comod}_{\mathcal{C}}$.

Motivations

Question

For a Hopf algebra H, classify the irreducible objects of ${}_H^H\mathcal{YD}$?

For general Hopf algebras H, this problem seems to be very difficult.
But for quasi-triangular (weak) Hopf algebras (H, R),...

1. Dijkgraaf, Pasquier, and Roche(1991): Present the structure theory of kG-YD-modules;
2. Reshetikhin and Semenov-Tian-Shansky(1988): Illustrates that $D(H) \cong (H \otimes H)^{\sigma}$ for a factorizable Hopf algebra H;
3. L. and Zhu(2019): Present the structure of YD-modules over semisimple and cosemisimple QT Hopf algebra.
4. For a QT weak Hopf algebras H,
   Liu G. and Zhu H.(2012): constructed a braided Hopf algebra B in $\mathcal{C} = {}_H\mathcal{M}$;
   Zhu H. and Zhang Y.(2015): prove that ${}_H^H\mathcal{YD} \cong B\text{-Comod}_{\mathcal{C}}$.

Categorical

Observations

1. kG is the group algebra, then $kG \in {}^{kG}_{kG}\mathcal{YD}$. Let $\{\mathcal{C}_i \mid 1 \leq i \leq r\}$ be the conjugacy classes of G. Then
$$kG = k\mathcal{C}_1 \oplus \cdots \oplus k\mathcal{C}_r$$
is a sum of simple YD-submodules.

Observations

1. kG is the group algebra, then $kG \in {}^{kG}_{kG}\mathcal{YD}$. Let $\{\mathcal{C}_i \mid 1 \leq i \leq r\}$ be the conjugacy classes of G. Then
$$kG = k\mathcal{C}_1 \oplus \cdots \oplus k\mathcal{C}_r$$
is a sum of simple YD-submodules.

2. $H \in {}^{H}_{H}\mathcal{YD}$ with left H-coaction via Δ and a left H-action via $\cdot_{ad}$.
Let $H = D_1 \oplus \cdots \oplus D_r$ be the decomposition of irreducible YD submodules. This leads to a decomposition of subcategories
$$ {}^{H}_{H}\mathcal{YD} \simeq {}^{H_R}_{H}\mathcal{M} = {}^{D_1}_{H}\mathcal{M} \oplus \cdots \oplus {}^{D_r}_{H}\mathcal{M}.$$

Observations

1. kG is the group algebra, then $kG \in {}^{kG}_{kG}\mathcal{YD}$. Let $\{\mathcal{C}_i \mid 1 \leq i \leq r\}$ be the conjugacy classes of G. Then
$$kG = k\mathcal{C}_1 \oplus \cdots \oplus k\mathcal{C}_r$$
is a sum of simple YD-submodules.
2. $H \in {}^{H}_{H}\mathcal{YD}$ with left H-coaction via Δ and a left H-action via $\cdot_{ad}$.
Let $H = D_1 \oplus \cdots \oplus D_r$ be the decomposition of irreducible YD submodules. This leads to a decomposition of subcategories
$${}^{H}_{H}\mathcal{YD} \simeq {}^{H_R}_{H}\mathcal{M} = {}^{D_1}_{H}\mathcal{M} \oplus \cdots \oplus {}^{D_r}_{H}\mathcal{M}.$$

Question

What about these for the Drinfeld center $\mathcal{Z}_l(\mathcal{C})$ of a braided multitensor category $\mathcal{C}$?

Reconstruction Theorem for Braided Tensor Categories (Majid)

1. Classical Reconstruction (Saavedra-Rivano; Deligne, Milne)
   Question: H – f.d. Hopf algebra, can we reconstruct H from ${}_H\mathcal{M}$?
   Yes. $\mathcal{C}$ – finite tensor category, $F : \mathcal{C} \to \mathrm{Vec}_k$ fiber functor, then $\mathrm{End}\,(F)$ is a Hopf algebra.

Reconstruction Theorem for Braided Tensor Categories (Majid)

1. Classical Reconstruction (Saavedra-Rivano; Deligne, Milne)
   Question: H – f.d. Hopf algebra, can we reconstruct H from ${}_H\mathcal{M}$?
   Yes. $\mathcal{C}$ – finite tensor category, $F : \mathcal{C} \to \mathrm{Vec}_k$ fiber functor, then $\mathrm{End}\,(F)$ is a Hopf algebra.
2. Are all finite tensor category of this form?
   finite tensor category $\longleftrightarrow$ f.d. weak quasi Hopf algebra (Haring-Oldenburg 1997)
   multifusion category $\longleftrightarrow$ semisimple weak Hopf algebra (Hayashi 1997; Szlachanyi 2001)

Reconstruction Theorem for Braided Tensor Categories (Majid)

1. Classical Reconstruction (Saavedra-Rivano; Deligne, Milne)
   Question: H – f.d. Hopf algebra, can we reconstruct H from ${}_H\mathcal{M}$?
   Yes. $\mathcal{C}$ – finite tensor category, $F : \mathcal{C} \to \mathrm{Vec}_k$ fiber functor, then $\mathrm{End}\,(F)$ is a Hopf algebra.
2. Are all finite tensor category of this form?
   finite tensor category $\longleftrightarrow$ f.d. weak quasi Hopf algebra (Haring-Oldenburg 1997)
   multifusion category $\longleftrightarrow$ semisimple weak Hopf algebra (Hayashi 1997; Szlachanyi 2001)

Braided Reconstruction(Majid,1991)

Let $\mathcal{C}$, $\mathcal{V}$ be braided tensor categories and $F : \mathcal{C} \to \mathcal{V}$ braided tensor functor. Then under some representability assumption there exists a Hopf algebra B in $\mathcal{V}$, such that F factor though the forgetful functor $B\text{-}\mathrm{Mod}_{\mathcal{V}} \to \mathcal{V}$.

Automorphism Braided Group

- (Majid,1991)Given a braided multitensor category $\mathcal{C}$ with *representability assumption for modules*, then the automorphism braided group of $\mathcal{C}$ is a $\mathcal{C}$-Hopf algebra B reconstructed from $\mathcal{C}$ by using the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

Automorphism Braided Group

- (Majid,1991)Given a braided multitensor category $\mathcal{C}$ with *representability assumption for modules*, then the automorphism braided group of $\mathcal{C}$ is a $\mathcal{C}$-Hopf algebra B reconstructed from $\mathcal{C}$ by using the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

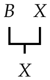
Representability assumption: there exist an object $B \in \mathcal{C}$ and a natural isomorphism $\theta_V : \mathrm{Hom}_{\mathcal{C}}(V, B) \to \mathrm{Nat}(V \otimes id_{\mathcal{C}}, id_{\mathcal{C}})$, and the maps

$$\theta_V^2 : \mathrm{Hom}_{\mathcal{C}}(V, B \otimes B) \to \mathrm{Nat}\left(V \otimes id_{\mathcal{C}}{}^{\otimes 2}, id_{\mathcal{C}}{}^{\otimes 2}\right),$$
$$\theta_V^3 : \mathrm{Hom}_{\mathcal{C}}(V, B \otimes B \otimes B) \to \mathrm{Nat}\left(V \otimes id_{\mathcal{C}}{}^{\otimes 3}, id_{\mathcal{C}}{}^{\otimes 3}\right),$$

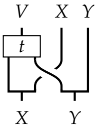
induced by $\alpha = \theta_B(id_B)$ and the braiding c, are bijective.

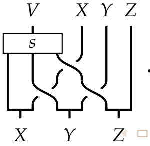
$$\begin{array}{ccc} \mathcal{C} & \to & \mathrm{Vec}_k \\ V & \mapsto & \mathrm{Nat}(V \otimes id_{\mathcal{C}}, id_{\mathcal{C}}) \end{array}$$

is representable.

With graphical convention, denote $\alpha_X =$ 

, then

$\theta_V^2(t)_{X,Y} =$ 

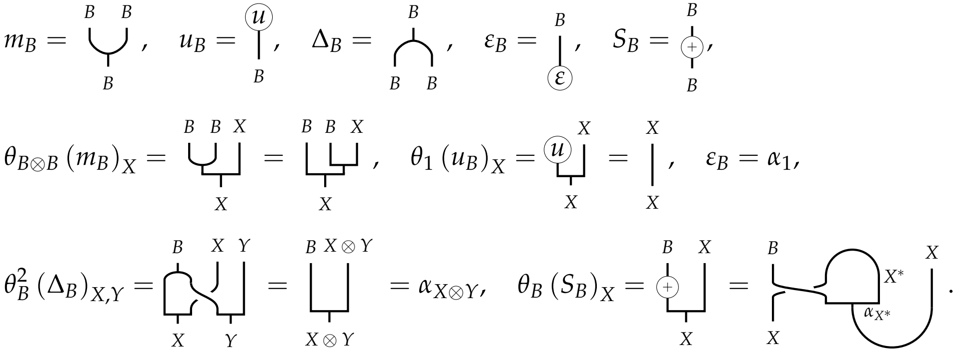
, $\theta_V^3(s)_{X,Y,Z} =$ 

.

Automorphism Braided Group

- (Majid,1991)Given a braided multitensor category $\mathcal{C}$ with *representability assumption for modules*, then the automorphism braided group of $\mathcal{C}$ is a $\mathcal{C}$-Hopf algebra B reconstructed from $\mathcal{C}$ by using the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

There is a $\mathcal{C}$-Hopf algebra structure on B, determined by the below diagrams :



The braided Hopf algebra B is called the automorphism braided group of $\mathcal{C}$.

Theorem (L., Zhu)

Let $\mathcal{C}$ be a finite braided multitensor, and B be the automorphism braided group of $\mathcal{C}$. Then the categories $\mathcal{Z}_l(\mathcal{C})$ and $B\text{-}\mathrm{Comod}_{\mathcal{C}}$ are isomorphic.

Theorem (L., Zhu)

Let $\mathcal{C}$ be a finite braided multitensor, and B be the automorphism braided group of $\mathcal{C}$. Then the categories $\mathcal{Z}_l(\mathcal{C})$ and $B\text{-}\mathrm{Comod}_{\mathcal{C}}$ are isomorphic.

Proof sketch

For any $M \in \mathcal{C}$, We have an isomorphism

$$\varphi_{M,M} : \mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M)$$

defined by the following composition

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(M \otimes {}^*M, B) &\xrightarrow{\theta_{M \otimes {}^*M}} \mathrm{Nat}(M \otimes {}^*M \otimes id_{\mathcal{C}}, id_{\mathcal{C}}) \\ &\xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M), \end{aligned}$$

Theorem (L., Zhu)

Let $\mathcal{C}$ be a finite braided multitensor, and B be the automorphism braided group of $\mathcal{C}$. Then the categories $\mathcal{Z}_l(\mathcal{C})$ and $B\text{-}\mathrm{Comod}_{\mathcal{C}}$ are isomorphic.

Proof sketch

For any $M \in \mathcal{C}$, We have an isomorphism

$$\varphi_{M,M} : \mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M)$$

defined by the following composition

$$\mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(M \otimes {}^*M, B) \xrightarrow{\theta_{M \otimes {}^*M}} \mathrm{Nat}(M \otimes {}^*M \otimes id_{\mathcal{C}}, id_{\mathcal{C}})$$
$$\xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M),$$

$$\begin{aligned} B\text{-}\mathrm{Comod}_{\mathcal{C}} &\to \mathcal{Z}_l(\mathcal{C}), \\ (M, \rho_M) &\mapsto \left(M, \varphi_{M,M}(\rho_M)\right), \end{aligned}$$

Theorem (L., Zhu)

Let $\mathcal{C}$ be a finite braided multitensor, and B be the automorphism braided group of $\mathcal{C}$. Then the categories $\mathcal{Z}_l(\mathcal{C})$ and B-$\mathrm{Comod}_{\mathcal{C}}$ are isomorphic.

Proof sketch

For any $M \in \mathcal{C}$, We have an isomorphism

$$\varphi_{M,M} : \mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M)$$

defined by the following composition

$$\mathrm{Hom}_{\mathcal{C}}(M, B \otimes M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(M \otimes {}^*M, B) \xrightarrow{\theta_{M \otimes {}^*M}} \mathrm{Nat}(M \otimes {}^*M \otimes id_{\mathcal{C}}, id_{\mathcal{C}})$$
$$\xrightarrow{\cong} \mathrm{Nat}(M \otimes id_{\mathcal{C}}, id_{\mathcal{C}} \otimes M),$$

$$\begin{aligned} B\text{-}\mathrm{Comod}_{\mathcal{C}} &\rightarrow \mathcal{Z}_l(\mathcal{C}), \\ (M, \rho_M) &\mapsto \left(M, \varphi_{M,M}(\rho_M)\right), \\ \mathcal{Z}_l(\mathcal{C}) &\rightarrow B\text{-}\mathrm{Comod}_{\mathcal{C}}, \\ (M, \gamma_{M,\bullet}) &\mapsto \left(M, \varphi^{-1}_{M,M}(\gamma_{M,\bullet})\right) \end{aligned}$$

Application to QT weak Hopf algebra

Theorem (L., Zhu)

Let $\mathcal{C}$ be a finite braided multitensor, and B be the automorphism braided group of $\mathcal{C}$. Then the categories $\mathcal{Z}_l(\mathcal{C})$ and $B\text{-}\mathrm{Comod}_{\mathcal{C}}$ are isomorphic.

Theorem

1. *(Liu G., Zhu H.,2012) Let (H,R) be a quasi-triangular weak Hopf algebra, and $\mathcal{C} = {}_H\mathcal{M}$. Then the object $B = (C_H(H_s), \cdot_{ad})$ is a Hopf algebra in $\mathcal{C}$, with structure as follows,*
    1. *multiplication $m_B : B \otimes_t B \to B$ and the unit $u_B : H_t \to B$ are defined by*

    $$m_B\left(1_{(1)} \cdot_{ad} a \otimes 1_{(2)} \cdot_{ad} b\right) = ab, \; u_B(z) = z, \quad \forall a,b \in B, z \in H_t.$$

    2. *comultiplication $\Delta_B : B \to B \otimes_t B$, counit $\varepsilon_B : B \to H_t$ and antipode $S_B : B \to B$ are defined by*

    $$\Delta_B(b) = b_{(1)} S\left(R^2\right) \otimes R^1 \cdot_{ad} b_{(2)}, \; \varepsilon_B(b) = \varepsilon_t(b), \; S_B(b) = R^2 S\left(R^1 \cdot_{ad} b\right).$$

2. *(L., Zhu) The braided Hopf algebra above is exactly the automorphism braided group of ${}_H\mathcal{M}$.*
3. *(Zhu H., Zhang Y.,2015)The category $B\text{-}\mathrm{Comod}_{\mathcal{C}}$ is isomorphic to ${}_H^H\mathcal{YD}$.*

Module categories over multitensor category

Definition (Bernstein, Crane-Frenkel)

Let $\mathcal{C}$ be a multitensor category. A right $\mathcal{C}$-module category is a collection $(\mathcal{M}, \otimes, a, r)$, where

- $\mathcal{M}$ *is a locally finite abelian category,*
- $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ *is a bifunctor, bilinear on morphisms and exact in the second variable,*
- *natrural isomorphisms:* $a_{M,X,Y} : (M \otimes X) \otimes Y \to M \otimes (X \otimes Y)$, $r_M : M \otimes 1 \to M$, *surject to pentagon axiom and triangular axiom.*

Module categories over multitensor category

Definition (Bernstein, Crane-Frenkel)

Let $\mathcal{C}$ be a multitensor category. A right $\mathcal{C}$-module category is a collection $(\mathcal{M}, \otimes, a, r)$, where

- *$\mathcal{M}$ is a locally finite abelian category,*
- *$\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is a bifunctor, bilinear on morphisms and exact in the second variable,*
- *natrural isomorphisms: $a_{M,X,Y} : (M \otimes X) \otimes Y \to M \otimes (X \otimes Y)$, $r_M : M \otimes 1 \to M$, surject to pentagon axiom and triangular axiom.*

Examples

1. *$\mathcal{C}$ is a right $\mathcal{C}$-module category.*
2. *Let D be a coalgebra in $\mathcal{C}$. Then $D\text{-Comod}_{\mathcal{C}}$ is a right $\mathcal{C}$-module category.*
3. *If $\mathcal{C}$ is braided, then $\mathcal{Z}_l(\mathcal{C})$ is a right $\mathcal{C}$-module category via the tensor functor $\mathcal{C} \to \mathcal{Z}_l(\mathcal{C})$, $X \mapsto (X, c_{X,\bullet})$.*

Module categories over multitensor category

Definition (Bernstein, Crane-Frenkel)

Let $\mathcal{C}$ be a multitensor category. A right $\mathcal{C}$-module category is a collection $(\mathcal{M}, \otimes, a, r)$, where

- *$\mathcal{M}$ is a locally finite abelian category,*
- *$\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is a bifunctor, bilinear on morphisms and exact in the second variable,*
- *natrural isomorphisms: $a_{M,X,Y} : (M \otimes X) \otimes Y \to M \otimes (X \otimes Y)$, $r_M : M \otimes 1 \to M$, surject to pentagon axiom and triangular axiom.*

Examples

1. *$\mathcal{C}$ is a right $\mathcal{C}$-module category.*
2. *Let D be a coalgebra in $\mathcal{C}$. Then $D\text{-Comod}_{\mathcal{C}}$ is a right $\mathcal{C}$-module category.*
3. *If $\mathcal{C}$ is braided, then $\mathcal{Z}_l(\mathcal{C})$ is a right $\mathcal{C}$-module category via the tensor functor $\mathcal{C} \to \mathcal{Z}_l(\mathcal{C})$, $X \mapsto (X, c_{X,\bullet})$.*

- (L, Zhu) As right $\mathcal{C}$-module categories, $\mathcal{Z}_l(\mathcal{C}) \cong B\text{-Comod}_{\mathcal{C}}$.

$\mathcal{C}$-Cocommutative coalgebra

The automorphism braided group B is $\mathcal{C}$-cocommutative in the sense that for every object $X \in \mathcal{C}$, the B-action α_X on X satisfies the following identity

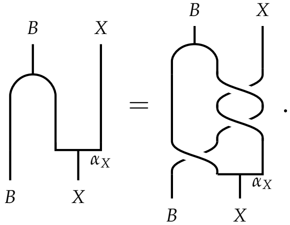
$$(id_B \otimes \alpha_X)(\Delta_B \otimes id_X) = (id_B \otimes \alpha_X)(c_{B,B} \otimes id_X)(id_B \otimes c_{X,B}c_{B,X})(\Delta_B \otimes id_X),$$

$\mathcal{C}$-Cocommutative coalgebra

The automorphism braided group B is $\mathcal{C}$-cocommutative in the sense that for every object $X \in \mathcal{C}$, the B-action α_X on X satisfies the following identity

$$(id_B \otimes \alpha_X)(\Delta_B \otimes id_X) = (id_B \otimes \alpha_X)(c_{B,B} \otimes id_X)(id_B \otimes c_{X,B}c_{B,X})(\Delta_B \otimes id_X),$$

that is,

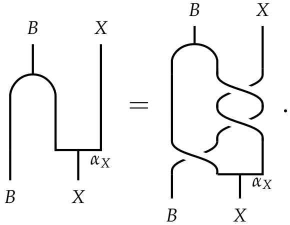


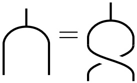
$\mathcal{C}$-Cocommutative coalgebra

The automorphism braided group B is $\mathcal{C}$-cocommutative in the sense that for every object $X \in \mathcal{C}$, the B-action α_X on X satisfies the following identity

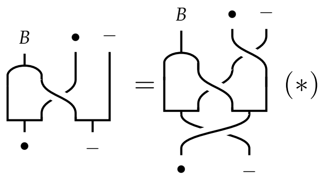
$$(id_B \otimes \alpha_X)(\Delta_B \otimes id_X) = (id_B \otimes \alpha_X)(c_{B,B} \otimes id_X)(id_B \otimes c_{X,B}c_{B,X})(\Delta_B \otimes id_X),$$

that is,

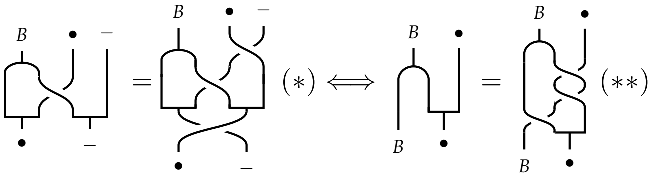


In general, it does not imply  .

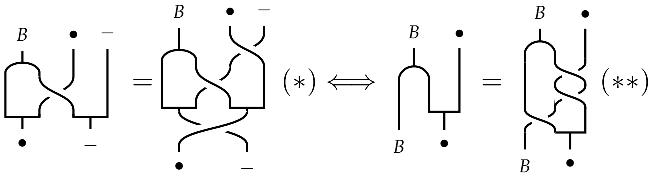
An interpretation of $\mathcal{C}$-cocommutative

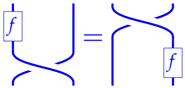


An interpretation of $\mathcal{C}$-cocommutative

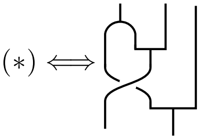


An interpretation of $\mathcal{C}$-cocommutative

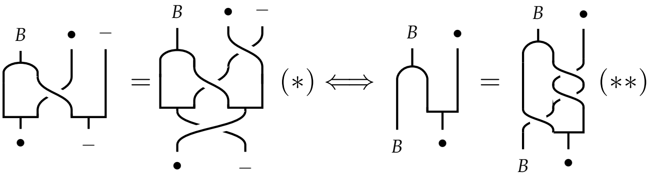




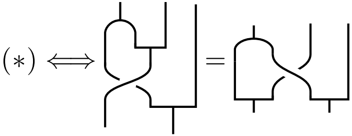
Proof:



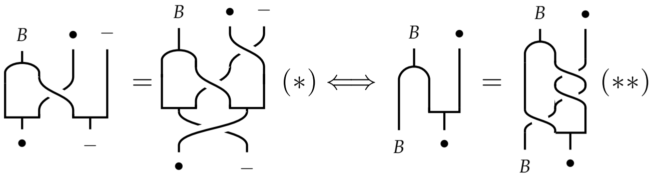
An interpretation of $\mathcal{C}$-cocommutative



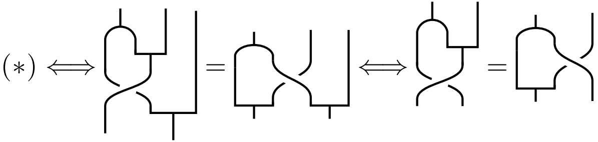
Proof:



An interpretation of $\mathcal{C}$-cocommutative

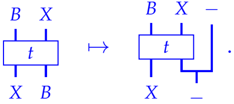


Proof:

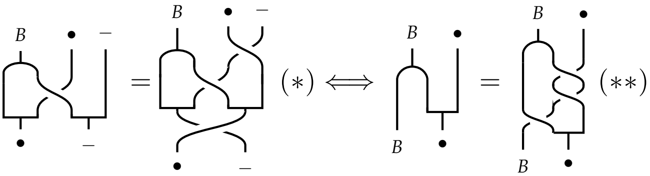


The following map induced by α is an isomorphism ,

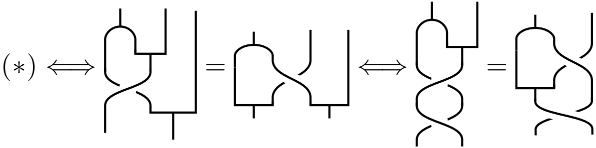
$$\mathrm{Hom}_{\mathcal{C}}\left(B \otimes X, X \otimes B\right) \quad \cong \quad \mathrm{Nat}\left(B \otimes X \otimes -, X \otimes -\right)$$



An interpretation of $\mathcal{C}$-cocommutative

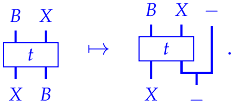


Proof:

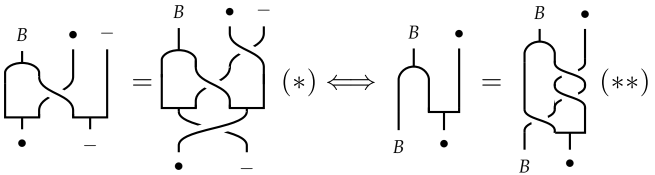


The following map induced by α is an isomorphism ,

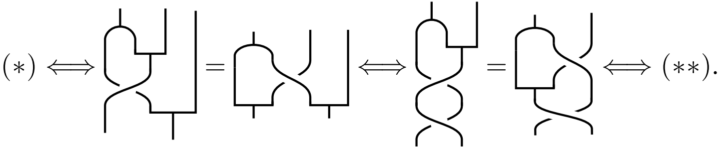
$$\mathrm{Hom}_{\mathcal{C}}(B \otimes X, X \otimes B) \quad \cong \quad \mathrm{Nat}(B \otimes X \otimes -, X \otimes -)$$



An interpretation of $\mathcal{C}$-cocommutative

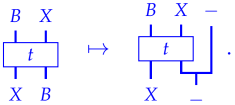


Proof:



The following map induced by α is an isomorphism ,

$$\mathrm{Hom}_{\mathcal{C}}(B \otimes X, X \otimes B) \quad \cong \quad \mathrm{Nat}(B \otimes X \otimes -, X \otimes -)$$



A Decomposition Theorem

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.
Theorem(L., Zhu) Let $i : (D, \rho_D) \to (B, \Delta_B)$ be a subobject of $B \in B\text{-}\mathrm{Comod}_{\mathcal{C}}$. Then

A Decomposition Theorem

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.
Theorem(L., Zhu) Let $i : (D, \rho_D) \to (B, \Delta_B)$ be a subobject of $B \in B\text{-Comod}_{\mathcal{C}}$. Then

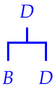
- there exists a unique $\mathcal{C}$-coalgebra structure on D such that i is a coalgebra morphism (i.e., D is a subcoalgebra of B),

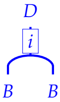
A Decomposition Theorem

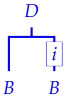
Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.

Theorem(L., Zhu) Let $i : (D, \rho_D) \to (B, \Delta_B)$ be a subobject of $B \in B\text{-Comod}_{\mathcal{C}}$. Then

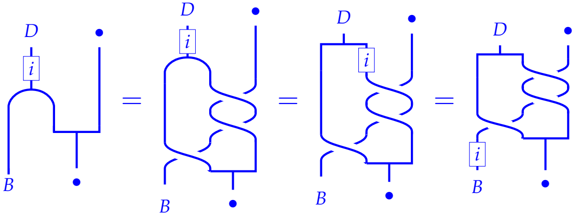
- there exists a unique $\mathcal{C}$-coalgebra structure on D such that i is a coalgebra morphism (i.e., D is a subcoalgebra of B),

Denoted $\rho_D =$ 

, then 

$=$ 

. On the other hand,



A Decomposition Theorem

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.

Theorem(L., Zhu) Let $i : (D, \rho_D) \to (B, \Delta_B)$ be a subobject of $B \in B\text{-}\mathrm{Comod}_{\mathcal{C}}$. Then

- there exists a unique $\mathcal{C}$-coalgebra structure on D such that i is a coalgebra morphism (i.e., D is a subcoalgebra of B),
- the category $D\text{-}\mathrm{Comod}_{\mathcal{C}}$ is a $\mathcal{C}$-module subcategory of $B\text{-}\mathrm{Comod}_{\mathcal{C}}$.

A Decomposition Theorem

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.
Theorem(L., Zhu) Let $i : (D, \rho_D) \to (B, \Delta_B)$ be a subobject of $B \in B\text{-}\mathrm{Comod}_{\mathcal{C}}$. Then

- there exists a unique $\mathcal{C}$-coalgebra structure on D such that i is a coalgebra morphism (i.e., D is a subcoalgebra of B),
- the category $D\text{-}\mathrm{Comod}_{\mathcal{C}}$ is a $\mathcal{C}$-module subcategory of $B\text{-}\mathrm{Comod}_{\mathcal{C}}$.

Moreover, the following statements are equivalent.

1. D is indecomposable in $B\text{-}\mathrm{Comod}_{\mathcal{C}}$.
2. D is indecomposable in $D\text{-}\mathrm{Comod}_{\mathcal{C}}$.
3. D is an indecomposable $\mathcal{C}$-coalgebra.
4. The $\mathcal{C}$-module category $D\text{-}\mathrm{Comod}_{\mathcal{C}}$ is indecomposable.

A Decomposition Theorem II

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.

Theorem

As an object of $\mathcal{Z}_l(\mathcal{C}) = B\text{-}\mathrm{Comod}_{\mathcal{C}}$, write $B = B_1 \oplus \cdots \oplus B_r$ as a direct sum of indecomposable subobjects.

1. *Then the decomposition $B = B_1 \oplus \cdots \oplus B_r$ is unique as a direct sum of indecomposable subobjects, and it is also unique as a direct sum of indecomposable $\mathcal{C}$-subcoalgebras.*
2. *The category $\mathcal{Z}_l(\mathcal{C})$ admits a unique decomposition*

$$\mathcal{Z}_l(\mathcal{C}) = B_1\text{-}\mathrm{Comod}_{\mathcal{C}} \oplus \cdots \oplus B_r\text{-}\mathrm{Comod}_{\mathcal{C}}$$

   into the direct sum of indecomposable $\mathcal{C}$-module subcategories.

Characterization of module categories

Let $\mathcal{M}$ be a semisimple left module category over a finite multitensor category $\mathcal{C}$.

Internel Hom: For any $M_1, M_2 \in \mathcal{M}$, the *internal Hom* $\underline{\mathrm{Hom}}(M_1, M_2)$ is an object of $\mathcal{C}$ representing the functor $X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2) : \mathcal{C} \to \mathrm{Vec}_k$, i.e. there exists a natural isomorphism

$$\eta_{\bullet, M_1, M_2} : \mathrm{Hom}_{\mathcal{M}}(\bullet \otimes M_1, M_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(\bullet, \underline{\mathrm{Hom}}(M_1, M_2)).$$

Theorem (Eingof, Ostrik, 2004)

If $M \in \mathcal{M}$ is a generator, then $A = \underline{\mathrm{Hom}}(M, M)$ is a semisimple algebra in $\mathcal{C}$. The functor $F = \underline{\mathrm{Hom}}(M, \bullet) : \mathcal{M} \to \mathrm{Mod}_{\mathcal{C}} - A$ given by $V \mapsto \underline{\mathrm{Hom}}(M, V)$ is an equivalence of $\mathcal{C}$-module categories.

Theorem (L., Zhu, 2019)

The functor $G = \bullet \otimes_A M : \mathrm{Mod}_{\mathcal{C}}(A) \to \mathcal{M}$ is a quasi-inverse to the equivalence $F = \underline{\mathrm{Hom}}(M, \bullet) : \mathcal{M} \to \mathrm{Mod}_{\mathcal{C}}(A)$.

Remark: If assume further that $\mathcal{M}$ is indecomposable, then every nonzero object M generates $\mathcal{M}$.

Internal Hom for $D\text{-Comod}_{\mathcal{C}}$

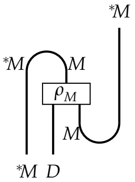
Proposition

Let D be a coalgebra in a multitensor category $\mathcal{C}$. Consider $D\text{-Comod}_{\mathcal{C}}$ as right $\mathcal{C}$-module category. Then for $M, N \in D\text{-Comod}_{\mathcal{C}}$, $\underline{\mathrm{Hom}}(M,N) = {}^*M \square_D^{\mathcal{C}} N$, i.e., the functor ${}^*M \square_D^{\mathcal{C}} \bullet : D\text{-Comod}_{\mathcal{C}} \to \mathcal{C}$ is a right adjoint of $M \otimes \bullet$.

Let $(M, \rho_M) \in D\text{-Comod}_{\mathcal{C}}$. Then ${}^*M$ has a natural right D-comodule structure $\rho_{{}^*M}$, which is the image of ρ_M under the composition of the isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(M, D \otimes M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(D^* \otimes M, M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}({}^*M, {}^*M \otimes D).$$

$\rho_{{}^*M} =$



.

Remark: Let $\mathcal{C} = {}_H\mathcal{M}$, where H is a (weak) Hopf algebra, and let D be an H-module coalgebra, then $\underline{\mathrm{Hom}}(M,N) = \mathrm{Hom}^D(M,N)$, for any $M, N \in D\text{-Comod}_{\mathcal{C}}$.

Let $\mathcal{C}$ be a braided finite multitensor category, and B be the automorphism braided group of $\mathcal{C}$.

Theorem

As an object of $\mathcal{Z}_l(\mathcal{C}) = B\text{-}\mathrm{Comod}_{\mathcal{C}}$, write $B = B_1 \oplus \cdots \oplus B_r$ as a direct sum of indecomposable subobjects.

1. *Then the decomposition $B = B_1 \oplus \cdots \oplus B_r$ is unique as a direct sum of indecomposable subobjects, and it is also unique as a direct sum of indecomposable $\mathcal{C}$-subcoalgebras.*
2. *The category $\mathcal{Z}_l(\mathcal{C})$ admits a unique decomposition*
$$\mathcal{Z}_l(\mathcal{C}) = B_1\text{-}\mathrm{Comod}_{\mathcal{C}} \oplus \cdots \oplus B_r\text{-}\mathrm{Comod}_{\mathcal{C}}$$
into the direct sum of indecomposable $\mathcal{C}$-module subcategories.
3. *For each $1 \leq i \leq r$, let $M_i \in B_i\text{-}\mathrm{Comod}_{\mathcal{C}}$ be a nonzero object such that the functor $F_i = {}^*M_i \Box^{\mathcal{C}}_{B_i} \bullet : B_i\text{-}\mathrm{Comod}_{\mathcal{C}} \to A_i\text{-}\mathrm{Mod}_{\mathcal{C}}$ is right exact, where $A_i = {}^*M_i \Box^{\mathcal{C}}_{B_i} M_i$. Then F_i is an equivalence between $\mathcal{C}$-module categories $B_i\text{-}\mathrm{Comod}_{\mathcal{C}}$ and $A_i\text{-}\mathrm{Mod}_{\mathcal{C}}$.*

YD module Structure theorem of Weak Hopf version

Let H be a semisimple quasi-triangular weak Hopf algebra, and B be the automorphism braided group of $\mathcal{C} = {}_H\mathcal{M}$. Let D be a minimal H-adjoint-stable subcoalgebra of B. Then the forgetful functor ${}_H^D\mathcal{M} \to {}^D\mathcal{M}$ has a left adjoint $\mathrm{Ind} : {}^D\mathcal{M} \to {}_H^D\mathcal{M}$.
Consider $D\text{-Comod}_{\mathcal{C}}$ as a module category over Vec_k, we get:

Theorem

Let D be a minimal H-adjoint-stable subcoalgebra of B. If $0 \neq W \in {}^D\mathcal{M}$, then $A_W = \mathrm{End}_H^D(\mathrm{Ind}(W))$ is a semisimple k-algebra, and the functors

$$F = \mathrm{Hom}_H^D(\mathrm{Ind}(W), \bullet) : D\text{-}\mathrm{Comod}_{\mathcal{C}} \to \mathcal{M}_{A_W},$$
$$G = \bullet \otimes_{A_W} (\mathrm{Ind}(W)) : \mathcal{M}_{A_W} \to D\text{-}\mathrm{Comod}_{\mathcal{C}}$$

establish an equivalence of between $D\text{-Comod}_{\mathcal{C}}$ and $\mathcal{M}_{A_W}$.

In some special cases, the results yield those appeared in

- Dijkgraaf, Pasquier, and Roche. Quasi-Hopf algebras, group cohomology and orbifold models, 1991).
- Gould. Quantum double finite group algebras and their representations, 1993).
- L and Zhu, On the structure of irreducible Yetter-Drinfeld modules over quasi-triangular Hopf algebras, 2019

YD module Structure theorem of Weak Hopf version

1. If $0 \neq W \in {}^{D}\mathcal{M}$, then $A_W = \mathrm{End}_H^D(\mathrm{Ind}(W))$ is a semisimple k-algebra, and the functors

$$F = \mathrm{Hom}_H^D(\mathrm{Ind}(W), \bullet) : D\text{-}\mathrm{Comod}_{\mathcal{C}} \to \mathcal{M}_{A_W},$$
$$G = \bullet \otimes_{A_W} (\mathrm{Ind}(W)) : \mathcal{M}_{A_W} \to D\text{-}\mathrm{Comod}_{\mathcal{C}}$$

   establish an equivalence of between D-$\mathrm{Comod}_{\mathcal{C}}$ and $\mathcal{M}_{A_W}$.

2. For any nonzero $M \in D$-$\mathrm{Comod}_{\mathcal{C}}$, the algebra $A = \mathrm{End}^D(M)^{op}$ in $\mathcal{C}$ is semisimple, and the functors

$$F = \mathrm{Hom}^D(M, \bullet) : D\text{-}\mathrm{Comod}_{\mathcal{C}} \to A\text{-}\mathrm{Mod}_{\mathcal{C}},$$
$$G = M \otimes_A \bullet : A\text{-}\mathrm{Mod}_{\mathcal{C}} \to D\text{-}\mathrm{Comod}_{\mathcal{C}}$$

   establish an equivalence of $\mathcal{C}$-module categories between D-$\mathrm{Comod}_{\mathcal{C}}$ and A-$\mathrm{Mod}_{\mathcal{C}}$.

Remark: Take $A'_W = \mathrm{End}^D(\mathrm{Ind}(W))^{op}$, then D-$\mathrm{Comod}_{\mathcal{C}} \cong A'_W$-$\mathrm{Mod}_{\mathcal{C}} \cong \mathcal{M}_{A_W}$.
In fact, as a k-algebra $A'_W \cong A_W \# H^*$. By dual theorem, $(A_W \# H^*) \# H$ is Morita equivalent to A_W.

Thank you!

References

S. Majid,
Braided groups and algebraic quantum field theories.
Letters in Mathematical Physics, 22(3):167–175, 1991.

V. Ostrik,
Module categories, weak Hopf algebras and modular invariants.
Transform. Groups, 8(2):177–206, 2003.

H. Zhu and Y. Zhang,
Braided autoequivalences and quantum commutative bi-Galois objects.
Journal of Pure and Applied Algebra, 219:4144–4167, 2015.

Z. Liu and S. Zhu,
On the structure of irreducible Yetter-Drinfeld modules over quasi-triangular Hopf algebras,
Journal of Algebra, 539:339-365, 2019.