Finite dual and Hopf pairing

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Outline

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1. Motivations
2. Determination of finite duals
3. Some consequences

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2. Determination of the finite duals
   - For infinite dihedral group algebra $\Bbbk\mathbb{D}_\infty$
   - For infinite dimensional Taft algebra $T_\infty(n, v, \xi)$
   - For generalized Liu algebra $B(n, \omega, \gamma)$
   - For the Hopf algebra $D(m, d, \xi)$
3. Some consequences

Preparation

- In this talk, $\Bbbk$ an algebraically closed field of characteristic zero.
- All spaces and algebras are over $\Bbbk$.
- This talk is based on the following works:

  (Joint with Ge Fan) A combinatorial identity and the finite dual of infinite dihedral group algebra. Mathematika 67 (2021) 498-513.
  (Joint with Kangqiao Li) The finite duals of affine prime regular Hopf algebras of GK-dimension one, arXiv 2103.00495.

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A Larson-Radford's result

- It is well-known that Larson-Radford (J. Algebra, 1988) proved the following result:

Theorem

Let H be a finite dimensional Hopf algebra, then H is semisimple if and only if H^* is semisimple.

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- Naively, the infinite dimensional analogue seems to be:
- A Hopf algebra H has finite global dimension if and only if H^* has finite global dimension?
- But H^* has no dual Hopf algebra structure in general.
- So a natural candidate for H^* is H°, the finite dual of H.

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A Takeuchi's definition

- Takeuchi defined a quantum group as follows.

Definition

A quantum group G is defined to be a triple

$$G = (A, U, \langle\ ,\ \rangle)$$

where A and U are Hopf algebras, and $\langle\ ,\ \rangle$ is a Hopf pairing on $U \times A$.

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Approach

- Basic idea: From H to nondegenerate Hopf pairing:
  $H \to (H, H^{\bullet})$.
- Raising many questions: Existence? Uniqueness?...
- Lucky point: We know the classification of some infinite-dimensional Hopf algebras, for example

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Classification

- Based on previous works, we already got a complete classification about noetherian prime regular Hopf algebras of GK-dim one:
- Infinite dimensional Taft algebra T_∞ ;
- Infinite dihedral group algebra $\mathbb{k}\mathbb{D}_\infty$;
- Generalized Liu's algebras $B(n, \omega, \gamma)$;
- Hopf algebras $D(m, d, \xi)$.
- In this talk, we will determine the finite duals of these Hopf algebras. From this, we test above questions.

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Finite Dual

- Let H be a Hopf algebra, the finite dual H° of H is defined by

$$H^\circ := \{f \in H^* | f(I) = 0, \text{some ideal } I \text{ s.t. } \dim(H/I) < \infty\}.$$

- A basic fact: H° is a Hopf algebra.

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Some known examples

Example

Let H_1 be the polynomial algebra $\Bbbk[x]$, $\Delta(x) = 1 \otimes x + x \otimes 1$. Then we have

$$H_1{}^\circ \cong \Bbbk[x] \otimes kG$$

where $G = (\Bbbk, +)$.

Example

Let H_2 be the infinite cyclic group algebra $\Bbbk[g, g^{-1}]$, $\Delta(g) = g \otimes g$. Then we have

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- There is a common point in above examples, that is, H is commutative. Therefore H° is cocommutative and thus one can apply Milnor-Moore's Theorem.

Example

Consider the quantum group $U_q(sl_n)$. Then we have

$$U_q(sl_n)^\circ \cong \mathcal{O}_q(SL_n)\#k\mathbb{Z}_2^{n-1}.$$

This is proved by Takeuchi in 1992.

- The key point of above example is that the category of finite-dimensional representations of $U_q(sl_n)$ is semisimple.

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Infinite dihedral group algebra $\Bbbk\mathbb{D}_\infty$

- **By definition,** the infinite dihedral group $\mathbb{D}_\infty$ is generated by two elements g and x satisfying

$$x^2 = 1, \quad xgx = g^{-1}.$$

- Note that $\Bbbk\mathbb{D}_\infty$ is not not commutative and thus $(\Bbbk\mathbb{D}_\infty)^\circ$ is not cocommutative. Also, the category of finite-dimensional representations of $\Bbbk\mathbb{D}_\infty$ is not semisimple.

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The Hopf algebra $k\mathbb{D}_\infty{}^\circ$

- As an algebra, $\Bbbk\mathbb{D}_\infty{}^\circ$ is generated by $E, \Phi_\lambda, \Psi_\lambda$ for $\lambda \in \Bbbk^* = \Bbbk \setminus \{0\}$ and subjects to the following relations

$$E\Phi_\lambda = \Phi_\lambda E, \quad E\Psi_\lambda = \Psi_\lambda E, \quad \Phi_1 = 1,$$

$$\Phi_{\lambda_1}\Psi_{\lambda_2} = \Psi_{\lambda_1}\Phi_{\lambda_2} = \Psi_{\lambda_1\lambda_2}, \quad \Phi_{\lambda_1}\Phi_{\lambda_2} = \Phi_{\lambda_1\lambda_2}, \quad \Psi_{\lambda_1}\Psi_{\lambda_2} = \Phi_{\lambda_1\lambda_2}$$

for all $\lambda, \lambda_1, \lambda_2 \in \Bbbk^*$.

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The Hopf algebra $\Bbbk\mathbb{D}_\infty^\circ$

The comultiplication, counit and the antipode are given by

$$\Delta(E) = E \otimes 1 + \psi_1 \otimes E,$$

$$\Delta(\Phi_\lambda) = \frac{1}{2}(\phi_\lambda + \psi_\lambda) \otimes \Phi_\lambda + \frac{1}{2}(\Phi_\lambda - \Psi_\lambda) \otimes \Phi_{\lambda^{-1}},$$

$$\Delta(\Psi_\lambda) = \frac{1}{2}(\Phi_\lambda + \Psi_\lambda) \otimes \Psi_\lambda - \frac{1}{2}(\Phi_\lambda - \Psi_\lambda) \otimes \Psi_{\lambda^{-1}},$$

$$\varepsilon(E) = 0, \quad \varepsilon(\Phi_\lambda) = \varepsilon(\Psi_\lambda) = 1,$$

$$S(E) = -\psi_1 E, \quad S(\Phi_\lambda) = \frac{1}{2}(\Phi_{\lambda^{-1}} + \Psi_{\lambda^{-1}}) + \frac{1}{2}(\Phi_\lambda - \Psi_\lambda),$$

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for $\lambda \in \Bbbk^*$.

Main result

Lemma

With operations defined above, $\Bbbk\mathbb{D}_{\infty^\circ}$ is a Hopf algebra.

Theorem

As Hopf algebras, we have

$$(\Bbbk\mathbb{D}_\infty)^\circ \cong \Bbbk\mathbb{D}_{\infty^\circ}.$$

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Sketch of the proof: Generators

- Clearly, $\{g^j x^k \mid j \in \mathbb{Z}, k = 0, 1\}$ is a basis of $\Bbbk\mathbb{D}_\infty$. Denote its dual basis by $f_{j,k}$.
- Construct:

$$e := \sum_{i\in\mathbb{Z}} i(f_{i,0} + f_{i,1}) \quad : \quad g^j x^k \mapsto j,$$

$$\phi_\lambda := \sum_{i\in\mathbb{Z}} \lambda^i(f_{i,0} + f_{i,1}) \quad : \quad g^j x^k \mapsto \lambda^j,$$

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Sketch of the proof: Generation property

- Key: As an algebra, $(\Bbbk\mathbb{D}_\infty)^\circ$ is generated by E, Φ_λ and Ψ_λ.
- Define a map

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- Key: As an algebra, $(\Bbbk\mathbb{D}_\infty)^\circ$ is generated by E, Φ_λ and Ψ_λ.
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$$\Theta : \Bbbk\mathbb{D}_{\infty^\circ} \to (\Bbbk\mathbb{D}_\infty)^\circ, \quad E \mapsto e, \ \Phi_\lambda \mapsto \phi_\lambda, \ \Psi_\lambda \mapsto \psi_\lambda, \quad (\lambda \in \Bbbk^*)$$

which gives the desired isomorphism.

Remarks

- The link-indecomposable component (Montgomery's sense) containing 1 is the Hopf subalgebra generated by E, Ψ_1 which can be described as follows

$$E\Psi_1 = \Psi_1 E, \quad \Psi_1^2 = 1,$$
$$\Delta(E) = E \otimes 1 + \Psi_1 \otimes E, \quad \Delta(\Psi_1) = \Psi_1 \otimes \Psi_1.$$

- This verifies the infinite-dimensional case of the theorem of Larson-Radford. In our subsequent computations, we will find that the infinite-dimensional analogue of Larson-Radford's theorem is not always true.

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Infinite dimensional Taft algebra $T_\infty(n, v, \xi)$

- Let n be a positive integer, $0 \leq v \leq n-1$, and ξ be a primitive nth root of 1.
- As an algebra, $T_\infty(n, v, \xi)$ is generated by g and x with relations

$$g^n = 1, \quad xg = \xi gx.$$

Then $T_\infty(n, v, \xi)$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g^v, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$
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- For simplicity we denote $m := \frac{n}{\gcd(n,v)}$.
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$$\psi_\lambda, \ \omega, \ E_2, \ E_1$$

for $\lambda \in \Bbbk$ and subjects to the following relations

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for all $\lambda, \lambda_1, \lambda_2 \in \Bbbk$.

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The comultiplication, counit and the antipode are given by

$$\Delta(\omega) = \omega \otimes \omega, \quad \Delta(E_1) = 1 \otimes E_1 + E_1 \otimes \omega,$$

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$$\Delta(\psi_\lambda) = \sum_{c=0}^{n/m-1} (\psi_{\lambda\xi^{mc}} \otimes \psi_\lambda \sigma_c)(1 \otimes 1 + \lambda \sum_{k=1}^{m-1} E_1^{[k]} \otimes \omega^k E_1^{[m-k]}),$$

$$\varepsilon(\omega) = \varepsilon(\psi_\lambda) = 1, \quad \varepsilon(E_1) = \varepsilon(E_2) = 0, \quad S(\omega) = \omega^{n-1},$$

$$S(E_1) = -\xi^{-v}\omega^{n-1}E_1, \quad S(E_2) = -E_2, \quad S(\psi_\lambda) = \sum_{c=0}^{n/m-1} \psi_{-\lambda\xi^{-mc}}\sigma_c,$$

for $\lambda \in \Bbbk$, where $E_1^{[k]} := E_1^k / k!_{\xi^v}$ and $\sigma_c := \frac{m}{n} \sum_{l=0}^{n/m-1} \xi^{-lmc} \omega^{lm}$.

Main result

Lemma

With operations defined above, $T_{\infty^\circ}(n, v, \xi)$ is a Hopf algebra.

Theorem

As Hopf algebras, we have

$$T_\infty(n, v, \xi)^\circ \cong T_{\infty^\circ}(n, v, \xi).$$

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- The link-indecomposable component (Montgomery's sense) containing 1 is the Hopf subalgebra generated by ω, E_2, E_1 which can be described as follows

$$\omega^n = 1, \quad E_1^m = 0,$$
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Generalized Liu algebra $B(n, \omega, \gamma)$

- Let n and ω be positive integers, and γ be a primitive nth root of 1.
- As an algebra, $B(n, \omega, \gamma)$ is generated by $x^{\pm 1}$, g and y with relations

$$\begin{cases} xx^{-1} = x^{-1}x = 1, \quad xg = gx, \quad xy = yx, \\ yg = \gamma gy, \\ y^n = 1 - x^\omega = 1 - g^n. \end{cases}$$

Then $B(n, \omega, \gamma)$ becomes a Hopf algebra with comultiplication, counit and antipode are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = 1 \otimes y + y \otimes g,$$
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Main result

- Similarly, we construct a Hopf algebra $B_\circ(n,\omega,\gamma)$, and prove that

Theorem

As Hopf algebras, we have

$$B(n,\omega,\gamma)^\circ \cong B_\circ(n,\omega,\gamma).$$

Generators for $B(n, \omega, \gamma)^\circ$

- $\{x^i g^j y^l \mid 0 \leq i \leq \omega - 1,\ j \in \mathbb{Z},\ 0 \leq l \leq n-1\}$ is a basis of $B(n, \omega, \gamma)$.
- Construct:

$$\psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}} : x^i g^j y^l \mapsto \delta_{l,0} \lambda^{\frac{i}{\omega}} \lambda^{\frac{j}{n}},$$

$$E_2 : x^i g^j y^l \mapsto \delta_{l,0}\left(\frac{i}{\omega} + \frac{j}{n}\right), \quad E_1 : x^i g^j y^l \mapsto \delta_{l,1}$$

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Generation for $B(n, \omega, \gamma)^\circ$

- As an algebra, $B(n, \omega, \gamma)^\circ$ is generated by $\psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}}$, E_2 and E_1 with relations

$$\psi_{\lambda_1^{\frac{1}{\omega}}, \lambda_1^{\frac{1}{n}}} \psi_{\lambda_2^{\frac{1}{\omega}}, \lambda_2^{\frac{1}{n}}} = \psi_{(\lambda_1^{\frac{1}{\omega}} \lambda_2^{\frac{1}{\omega}}), (\lambda_1^{\frac{1}{n}} \lambda_2^{\frac{1}{n}})}, \quad \psi_{1,1} = 1, \quad E_1^n = 0,$$

$$E_2 \psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}} = \psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}} E_2, \quad E_1 \psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}} = \lambda^{\frac{1}{n}} \psi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}} E_1,$$

$$E_1 E_2 = E_2 E_1 + \frac{1}{n} E_1$$

for all $\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}}, \lambda_1^{\frac{1}{\omega}}, \lambda_1^{\frac{1}{n}}, \lambda_2^{\frac{1}{\omega}}, \lambda_2^{\frac{1}{n}} \in \Bbbk^*$.

Generation for $B(n, \omega, \gamma)^\circ$

The comultiplication, counit and the antipode are given by

$$\Delta(E_1) = 1 \otimes E_1 + E_1 \otimes \psi_{1,\gamma},$$

$$\Delta(E_2) = 1 \otimes E_2 + E_2 \otimes 1 - \sum_{k=1}^{n-1} E_1^{[k]} \otimes \psi_{1,\gamma}^k E_1^{[n-k]},$$

$$\Delta(\psi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}}) = (\psi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}} \otimes \psi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}})(1 \otimes 1 + (1-\lambda)\sum_{k=1}^{n-1} E_1^{[k]} \otimes \psi_{1,\gamma}^k E_1^{[n-k]}),$$

$$\varepsilon(\psi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}}) = 1, \quad \varepsilon(E_1) = \varepsilon(E_2) = 0,$$

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for $\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{n}} \in \Bbbk^*$, where $E_1^{[k]} := E_1^k / k!_\xi$ for $1 \leq k \leq n-1$.

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The Hopf algebra $D(m, d, \xi)$

- Let n, d be positive integers such that $(1+m)d$ is even, and ξ be a primitive $2m$th root of 1. Define $\omega := md$, $\gamma := \xi^2$.
- As an algebra, $D(m, d, \xi)$ is generated by $x^{\pm 1}, g, y, u_0, u_1, \cdots, u_{m-1}$ with relations

$$xx^{-1} = x^{-1}x = 1, \quad gx = xg, \quad yx = xy,$$
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The Hopf algebra $D(m, d, \xi)$

$$u_i u_j = \begin{cases} (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\phi_{i+1}\cdots\phi_{m-2-j}y^{i+j}g \\ \hfill (i+j \leq m-2) \\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}y^{i+j}g \\ \hfill (i+j = m-1) \\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g \\ \hfill (i+j \geq m) \end{cases}$$

where $\phi_i := 1 - \gamma^{-i-1}x^d$ and $0 \leq i, j \leq m-1$.

Generators for $D(m, d, \xi)^\circ$

- $\{x^i g^j y^l, x^i g^j u_l \mid 0 \le i \le \omega - 1,\ j \in \mathbb{Z},\ 0 \le l \le m - 1\}$ is a basis of $D(m, d, \xi)$.
- Construct:

$$\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0}\lambda^{\frac{i}{\omega}}\lambda^{\frac{j}{m}} \\ x^i g^j u_l \mapsto 0 \end{cases},\quad \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto 0 \\ x^i g^j u_l \mapsto \delta_{l,0}\lambda^{\frac{i}{\omega}}\lambda^{\frac{j}{m}} \end{cases},$$

$$E_2 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0}(\frac{i}{\omega} + \frac{j}{m}) \\ x^i g^j u_l \mapsto \delta_{l,0}(\frac{i}{\omega} + \frac{j}{m}) \end{cases},\quad E_1 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,1} \\ x^i g^j u_l \mapsto \frac{\xi}{1-\gamma^{-1}}\delta_{l,1} \end{cases},$$

for any $\lambda \in \Bbbk^*$, where $\lambda^{\frac{1}{\omega}}$ and $\lambda^{\frac{1}{n}}$ denote arbitrary ωth and nth roots of λ respectively.

Generators for $D(m, d, \xi)^\circ$

- $\{x^i g^j y^l, x^i g^j u_l \mid 0 \leq i \leq \omega - 1,\ j \in \mathbb{Z},\ 0 \leq l \leq m-1\}$ is a basis of $D(m, d, \xi)$.
- Construct:

$$\zeta_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0} \lambda^{\frac{i}{\omega}} \lambda^{\frac{j}{m}} \\ x^i g^j u_l \mapsto 0 \end{cases}, \quad \chi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto 0 \\ x^i g^j u_l \mapsto \delta_{l,0} \lambda^{\frac{i}{\omega}} \lambda^{\frac{j}{m}} \end{cases},$$

$$E_2 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0}\left(\frac{i}{\omega} + \frac{j}{m}\right) \\ x^i g^j u_l \mapsto \delta_{l,0}\left(\frac{i}{\omega} + \frac{j}{m}\right) \end{cases}, \quad E_1 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,1} \\ x^i g^j u_l \mapsto \frac{\xi}{1-\gamma^{-1}} \delta_{l,1} \end{cases},$$

for any $\lambda \in \Bbbk^*$, where $\lambda^{\frac{1}{\omega}}$ and $\lambda^{\frac{1}{n}}$ denote arbitrary ωth and nth roots of λ respectively.

Generation for $D(m, d, \xi)^\circ$

As an algebra, $D(m, d, \xi)^\circ$ is generated by $\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}}$, $\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{n}}}$, E_2, E_1 with relations

$$\zeta_{\lambda_1^{\frac{1}{\omega}},\lambda_1^{\frac{1}{m}}}\zeta_{\lambda_2^{\frac{1}{\omega}},\lambda_2^{\frac{1}{m}}} = \zeta_{(\lambda_1^{\frac{1}{\omega}}\lambda_2^{\frac{1}{\omega}}),(\lambda_1^{\frac{1}{m}}\lambda_2^{\frac{1}{m}})}, \quad \chi_{\lambda_1^{\frac{1}{\omega}},\lambda_1^{\frac{1}{m}}}\chi_{\lambda_2^{\frac{1}{\omega}},\lambda_2^{\frac{1}{m}}} = \chi_{(\lambda_1^{\frac{1}{\omega}}\lambda_2^{\frac{1}{\omega}}),(\lambda_1^{\frac{1}{m}}\lambda_2^{\frac{1}{m}})},$$

$$\zeta_{\lambda_1^{\frac{1}{\omega}},\lambda_1^{\frac{1}{m}}}\chi_{\lambda_2^{\frac{1}{\omega}},\lambda_2^{\frac{1}{m}}} = \chi_{\lambda_1^{\frac{1}{\omega}},\lambda_1^{\frac{1}{m}}}\zeta_{\lambda_2^{\frac{1}{\omega}},\lambda_2^{\frac{1}{m}}} = 0, \quad E_1E_2 = E_2E_1 + \frac{1}{m}\zeta_{1,1}E_1,$$

$$E_2\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} = \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_2, \quad E_1\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} = \lambda^{\frac{1}{m}}\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1,$$

$$E_2\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} = \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_2, \quad E_1\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} = \lambda^{\frac{-d}{\omega}}\lambda^{\frac{1}{m}}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1,$$

$$\zeta_{1,1} + \chi_{1,1} = 1, \quad E_1^m = \frac{1}{(1-\gamma)^m}\chi_{1,1},$$

for $\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}}, \lambda_1^{\frac{1}{\omega}}, \lambda_1^{\frac{1}{m}}, \lambda_2^{\frac{1}{\omega}}, \lambda_2^{\frac{1}{m}} \in \Bbbk^*$.

Generation for $D(m, d, \xi)^\circ$

- Denote $E_1^{[k]} := E_1^k / k!_\gamma$ for $1 \leq k \leq m-1$. The comultiplication is given by:

$$\begin{aligned}
\Delta(E_1) &= 1 \otimes E_1 + E_1 \otimes (\zeta_{1,\gamma} + \xi\chi_{1,\gamma}),\\
\Delta(E_2) &= (\zeta_{1,1} - \chi_{1,1}) \otimes E_2 + E_2 \otimes 1\\
&\quad + \sum_{k=1}^{m-1} (\zeta_{1,1} - \chi_{1,1}) E_1^{[k]} \otimes (\zeta_{1,\gamma} + \xi\chi_{1,\gamma})^k E_1^{[m-k]}.
\end{aligned}$$

- We remark that $\zeta_{1,1} - \chi_{1,1} = (\zeta_{1,\gamma} + \xi\chi_{1,\gamma})^m$. Also,

$$\Delta(\zeta_{1,\gamma} + \xi\chi_{1,\gamma}) = (\zeta_{1,\gamma} + \xi\chi_{1,\gamma}) \otimes (\zeta_{1,\gamma} + \xi\chi_{1,\gamma}).$$

Generation for $D(m,d,\xi)^\circ$

Suppose $(\lambda^{\frac{1}{\omega}})^d=\lambda^{\frac{1}{m}}$. Then

$$\begin{aligned}
&\Delta(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})\\
=\ &\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\otimes\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}+(1-\lambda)\sum_{k=1}^{n-1}\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1^{[k]}\otimes\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\zeta_{1,\gamma}^kE_1^{[n-k]}\\
&+(1-\lambda)\lambda^{\frac{(1-m)d/2}{\omega}}\Big(\frac{1/m}{1-\lambda^{\frac{1}{m}}}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\otimes\chi_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}\\
&\qquad+\sum_{k=1}^{m-1}\frac{1-\gamma^{-k}}{1-\gamma^{-k}\lambda^{\frac{1}{m}}}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1^{[k]}\otimes\chi_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}\xi^k\chi_{1,\gamma}^kE_1^{[m-k]}\Big).
\end{aligned}$$

Generation for $D(m, d, \xi)^\circ$

Suppose $(\lambda^{\frac{1}{\omega}})^d = \lambda^{\frac{1}{m}}$. Then

$$\begin{aligned}
\Delta(\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}) &= \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \otimes \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} + \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \otimes \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \\
&-(1-\lambda^{\frac{1}{m}})[\sum_{k=1}^{m-1}(1-\gamma\lambda^{\frac{1}{m}})\cdots(1-\gamma^{k-1}\lambda^{\frac{1}{m}})(1-\gamma^k)\cdots(1-\gamma^{m-1}) \\
&\qquad \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} E_1^{[k]} \otimes \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \xi^k \chi_{1,\gamma}^k E_1^{[m-k]} \\
&+\sum_{k=1}^{m-1} \lambda^{\frac{k}{m}-1}(1-\gamma\lambda^{\frac{1}{m}})\cdots(1-\gamma^{m-k-1}\lambda^{\frac{1}{m}})(1-\gamma^{m-k})\cdots(1-\gamma^{m-1}) \\
&\qquad \chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} E_1^{[k]} \otimes \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \zeta_{1,\gamma}^k E_1^{[m-k]}].
\end{aligned}$$

Generation for $D(m, d, \xi)^\circ$

- The coproduct on $\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}$ for arbitrary $\lambda^{\frac{1}{\omega}}$ and $\lambda^{\frac{1}{m}}$ is defined as $\Delta(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{d}{\omega}}})\Delta(\zeta_{1,\gamma}+\xi\chi_{1,\gamma})^k$, where k is a non-negative integer such that $\lambda^{\frac{1}{m}}=\lambda^{\frac{d}{\omega}}\gamma^k$. Similar definition is given for $\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}$.
- The counit and antipode:

$$\varepsilon(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})=1,\ \ \varepsilon(\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})=0,\ \ \varepsilon(E_1)=\varepsilon(E_2)=0.$$

$$S(E_1)=-\gamma^{-1}(\zeta_{1,\gamma^{-1}}+\xi^{-1}\chi_{1,\gamma^{-1}})E_1,$$

$$S(E_2)=-\zeta_{1,1}E_2+\chi_{1,1}E_2+\frac{1-m}{2m}\chi_{1,1},$$

$$S(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})=\zeta_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}},$$

$$S(\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})=\lambda^{\frac{(1-m)d/2}{\omega}}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\quad\text{when}\quad(\lambda^{\frac{1}{\omega}})^d=\lambda^{\frac{1}{m}}.$$

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S(E_1) &= -\gamma^{-1}(\zeta_{1,\gamma^{-1}} + \xi^{-1}\chi_{1,\gamma^{-1}})E_1,\\
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S(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}) &= \zeta_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}},\\
S(\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}) &= \lambda^{\frac{(1-m)d/2}{\omega}}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \quad \text{when} \quad (\lambda^{\frac{1}{\omega}})^d = \lambda^{\frac{1}{m}}.
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Remarks

- The link-indecomposable component (Montgomery's sense) containing 1 is the Hopf subalgebra generated by $\zeta_{1,\gamma}, \chi_{1,\gamma}, E_2, E_1$.
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Outline

1. Motivations
2. Determination of the finite duals
   - For infinite dihedral group algebra $\Bbbk\mathbb{D}_\infty$
   - For infinite dimensional Taft algebra $T_\infty(n, v, \xi)$
   - For generalized Liu algebra $B(n, \omega, \gamma)$
   - For the Hopf algebra $D(m, d, \xi)$
3. Some consequences

Direct consequences

- Let H be a prime regular Hopf algebra of GK-dim one and $H^{\bullet}$ the link-indecomposable component containing 1 of H°. Then we have

Proposition

(1) *The Hopf algebra $H^{\bullet}$ has GK-dimension one.*

(2) *Hopf algebras $(\Bbbk\mathbb{D}_{\infty})^{\bullet}$, $T_{\infty}(n,v,\xi)^{\bullet}$, $B(n,\omega,\gamma)^{\bullet}$ and $D(m,d,\xi)^{\bullet}$ are all pointed.*

(3) *The Hopf algebra $(\Bbbk\mathbb{D}_{\infty})^{\bullet}$ is regular while $T_{\infty}(n,v,\xi)^{\bullet}$, $B(n,\omega,\gamma)^{\bullet}$ and $D(m,d,\xi)^{\bullet}$ are not when $n,m \geq 2$.*

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Remarks

- Naturally, $(H, H^{\bullet})$ is a nondegenerate Hopf pairing and thus a quantum group in the Takeuchi's sense.
- $H^{\bullet}$ is "unique" in the following sense: $H^{\bullet}$ has the same GK-dim as H and is minimal under containing relation.This might be a version negating the semisimplicity result by Larson and Radford in infinite-dimensional cases.
- For a prime regular Hopf algebra H of GK-dim one, one can find two nondegenerate Hopf pairings (H, H_1), (H, H_2) with $H_1 \ncong H_2$.

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Questions

- We pose several questions as follows:
- For a general infinite-dimensional Hopf algebra H which is residually finite-dimensional, when does a minimal Hopf algebra $H^{\bullet}$ forming a non-degenerate Hopf pairing over H exist?
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- Thanks for your attention!