Quantum (dual) Grassmann superalgebra as $\mathcal{U}_q(\mathfrak{gl}(m|n))$-module algebra and beyond

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Background and Contents

- As we known, there was a Majid "quantum tree"([Ma]) question for the Drinfeld-Jimbo type quantum groups $U_q(\mathfrak{g})$ of semisimple Lie algebras $\mathfrak{g}$. An important application of this theory is introduced in [Hu], in which $U_q(\mathfrak{g}(n))$ may be realized as a certain quantum differential operators defined over the corresponding quantum planes.
- As a generalization of Majid "quantum tree"([Ma]) question in super case Hu introduce the notation of quantum Grassmann superalgebra $\Omega_q(m|n)$ and dual quantum Grassmann superalgebra $\Lambda_q(m|n)$ by defining quantum affine $(m|n)$-superspace $A_q^{m|n}$ and its Manin dual superspace $(A_q^{m|n})^!$ in [FHZZ].
- Furthermore, $U_q(\mathfrak{g}(m|n))$ is realized as a certain quantum differential operators defined over quantum affine $(m|n)$-superspace $A_q^{m|n}$ and its Manin dual superspace $(A_q^{m|n})^!$, so the quantum Grassmann superalgebra $\Omega_q(m|n)$ and dual quantum Grassmann superalgebra $\Lambda_q(m|n)$ are made into $U_q(\mathfrak{g}(m|n))$-module superalgebra respectively.

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- Our work is based on a previous collaboration: The realization of characterization of the quantum general linear superalgebra $U_q(\mathfrak{gl}(m|n))$ on the quantum Grassmann superalgebra $\Omega_q(m|n)$ and its dual Grassmann superalgebra $\Lambda_q(m|n)$ as quantum differential operators.
(see: [FHZZ,arXiv:1909.10276])

- Based on this model, we solve some representation problems of the quantum general linear superalgebra at unit roots and non-unit roots. The complete reducibility of the quantum Grassmann superalgebra $\Omega_q(m|n)$ as $U_q(\mathfrak{gl}(m|n))$- module is determined when q is genetic; In the case of q is a root of unity, we describe the Loewy filtration of the indecomposable module $\Omega_q(m|n)$, prove its rigidity and determine its dimensions as some combination formulas of these simple modules.(see: [GH, J. Algebra, 15]).

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- The most important breakthrough in this work is the definition and construction of the super quantum de Rham complexe, especially the proper definitions and constructions of quantum differentials as module homomorphisms. We describe the structures of these cohomology modules and determine their dimension formulas(as some combination formulas).

- The significance of this work in representation theory lies in that when the quantum parameter q is generic,the de Rham cohomology complex is exact except for the first term, that is, the cohomology group of every order disappears; The nondisappearance of quantum de Rham cohomology group at root of unity case explicitly measures the complexity of the modular structures of the quantum general linear superalgebra in modular representation theory (compared with the general representation theory at generic case).

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Notation and Definitions

We work on an algebraically closed field $\mathbb{K}$ of characteristic 0. Denote by $\mathbb{Z}_+$, $\mathbb{N}$ the set of nonnegative integers, positive integers, respectively.

Definition(Yu.I.Manin, 88)

For $q \in \mathbb{K}^*$, **the quantum symmetric algebra** $\mathbb{K}[A_q^m]$ is defined as a $\mathbb{K}$-vector space,

$$\mathbb{K}[A_q^m] = \mathbb{K}\{x_1, \cdots, x_m\}/(x_j x_i - q x_i x_j, i < j).$$

Definition([Hu], 00)

The quantum divided power algebra $\mathcal{A}_q(m) := \langle x^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^m \rangle$ is defined as a $\mathbb{K}$- vector space with a monomial basis $x^{(\alpha)}$. The multiplication is given by

$$x^{(\alpha)} x^{(\beta)} = q^{\alpha * \beta} \begin{bmatrix} \alpha + \beta \\ \alpha \end{bmatrix} x^{(\alpha+\beta)},$$

where, $\begin{bmatrix} \alpha+\beta \\ \alpha \end{bmatrix} := \prod_{i=1}^{n} \begin{bmatrix} \alpha_i+\beta_i \\ \alpha_i \end{bmatrix}$ and $\begin{bmatrix} \alpha_i+\beta_i \\ \alpha_i \end{bmatrix} = \frac{[\alpha_i+\beta_i]!}{[\alpha_i]![\beta_i]!}$ for any $\alpha_i, \beta_i \in \mathbb{Z}_+$.

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For $1 \leq i, j \leq n$, **the quantum exterior algebra** $\Lambda_{q^{-1}}^n$ is defined as

$$\Lambda_{q^{-1}}^n = \mathbb{K}\{x_1, \cdots, x_n\}/(x_i^2, x_j x_i + q x_i x_j, i < j).$$ with a monomial basis $\{x^\mu \mid \mu \in \mathbb{Z}_+^n\}$, where $x^0 = 1$.

Definition[FHZZ,arXiv:1909.10276]

Consider algebraic structure on the tensor space $\Omega_q(m|n) := \mathcal{A}_q(m) \otimes_{\mathbb{K}} \Lambda_{q^{-1}}^n$, the **quantum Grassmann superalgebra** $\Omega_q(m|n)$ is defined. It is an associative $\mathbb{K}$-superalgebra defined on a superspace over $\mathbb{K}$ with the multiplication given by:

$$(x^{(\alpha)} \otimes x^\mu) \cdot (x^{(\beta)} \otimes x^\nu) = q^{\mu * \beta} x^{(\alpha)} x^{(\beta)} \otimes x^\mu x^\nu,$$ where, $x^{(\alpha)}, x^{(\beta)} \in \mathcal{A}_q(m), x^\mu, x^\nu \in \Lambda_{q^{-1}}(n)$.

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Loewy filtration of $\Omega_q(m|n, \mathbf{r})$ and its rigidity

For any $m, n \in \mathbb{Z}_+$, denote:
$I = \{1, 2, \cdots, m+n\}$; $J := \{1, 2, \cdots, m+n-1\}$;
$I_0 := \{1, 2, \cdots, m\}$; $I_1 := \{m+1, m+2, \cdots, m+n\}$;
then $I = I_0 \cup I_1$.Consider the $U_q(\mathfrak{gl}(m|n))$-module algebra structure over the quantum Grassmann superalgebra:

Proposition(FHZZ,arXiv:1909.10276)

For any $x^{(\alpha)} \otimes x^{\mu} \in \Omega_q(m|n)$, set
(i) When $i \in I_0 \setminus m$,

$$e_i(x^{(\alpha)} \otimes x^{\mu}) = [\alpha_i + 1] x^{(\alpha+\varepsilon_i-\varepsilon_{i+1})} \otimes x^{\mu},$$
$$f_i(x^{(\alpha)} \otimes x^{\mu}) = [\alpha_{i+1} + 1] x^{(\alpha-\varepsilon_i+\varepsilon_{i+1})} \otimes x^{\mu},$$
$$K_i(x^{(\alpha)} \otimes x^{\mu}) = q^{\alpha_i} x^{(\alpha)} \otimes x^{\mu}.$$

(ii) when $i = m$, $\cdots$
(iii) when $i \in I_1$, $\cdots$
where e_i, f_i, K_i,$1 \le i \le m+n$ are the generators of $U_q(\mathfrak{gl}(m|n))$.

Loewy filtration of $\Omega_q(m|n,\mathbf{r})$ and its rigidity

We will focus our discussion on the case when q is a unit root with $\textbf{char}(q) = l > 2$,
Let $\mathbf{r} = (rl-1, rl-1, \cdots, rl-1) \in \mathbb{Z}_+^m, r \in \mathbb{N}$,
Consider the truncated objects of $\mathcal{A}_q(m)$ and $\Omega_q(m|n)$ denoted by $\mathcal{A}_q(m,\mathbf{r})$ and $\Omega_q(m|n,\mathbf{r})$, respectively.

Definition

$$\mathcal{A}_q(m,\mathbf{r}) := \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)} \in \mathcal{A}_q(m) \mid \alpha \leq \mathbf{r}\} \subseteq \mathcal{A}_q(m),$$
$$\Omega_q(m|n,\mathbf{r}) := \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)} \otimes x^{\mu} \in \Omega_q(m|n) \mid \alpha \leq \mathbf{r}\} \subseteq \Omega_q(m|n).$$

Definition

$$\Omega_q^{(s)}(m|n,\mathbf{r}) := \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)} \otimes x^{\mu} \in \Omega_q(m|n) \mid |\langle\alpha,\mu\rangle| = s\},$$

then $\Omega_q(m|n,\mathbf{r}) = \bigoplus_{s=0}^{N} \Omega_q^{(s)}(m|n,\mathbf{r})$, where $N = |\mathbf{r}| + n = m(rl-1) + n$.

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We still focus our discussion on the case when $\mathbb{Q}(q) \subseteq \mathbb{K}$, $\mathbf{char}(q) = l > 2$, and $U = U_q(\mathfrak{gl}(m|n))$ with $m > 2$.

Definition

For any $x^{(\alpha)} \otimes x^{\mu} \in \Omega_q(m|n,\mathbf{r})$ or $\Omega_q(m|n)$, the **energy** of $x^{(\alpha)} \otimes x^{\mu}$ denoted by $\mathrm{Edeg} x^{(\alpha)} \otimes x^{\mu}$ is defined as:

$\mathrm{Edeg} x^{(\alpha)} \otimes x^{\mu} := \sum_{i=1}^{m} \lfloor \frac{\alpha_i}{l} \rfloor = \sum_{i=1}^{m} \mathrm{Edeg}_i x^{(\alpha)} \otimes x^{\mu}$, where

$\mathrm{Edeg}_i x^{(\alpha)} \otimes x^{\mu} := \lfloor \frac{\alpha_i}{l} \rfloor$, for $i \in I_0$. Generally, for any $x = \sum_{\alpha \in I} k_\alpha x^{(\alpha)} \otimes x^{\mu} \in \Omega_q(m|n,\mathbf{r})$ or $\Omega_q(m|n)$,

$\mathrm{Edeg} x := \max\{\mathrm{Edeg} x^{(\alpha)} \otimes x^{\mu} \mid \alpha \in I\}$.

Loewy filtration of $\Omega_q(m|n,\mathbf{r})$ and its rigidity

Proposition

$\mathrm{Edeg}(u\cdot x^{(\alpha)}\otimes x^{\mu})\leq \mathrm{Edeg}(x^{(\alpha)}\otimes x^{\mu})$, for any $u\in U_q(\mathfrak{gl}(m|n))$, $x^{(\alpha)}\otimes x^{\mu}\in \Omega_q^{(s)}(m|n,\mathbf{r})$ or $\Omega_q^{(s)}(m|n)$. In particular, for each j, $\mathrm{Edeg}_j(u\cdot x^{(\alpha)}\otimes x^{\mu})\leq \mathrm{Edeg}_j(x^{(\alpha)}\otimes x^{\mu})$.

Proposition

Let $x^{(\alpha)}\otimes x^{\mu}$ and $x^{(\beta)}\otimes x^{\nu}\in \Omega_q^{(s)}(m|n,\mathbf{r})$ or $\Omega_q^{(s)}(m|n)$ with $\mathrm{Edeg}x^{(\alpha)}\otimes x^{\mu}=\mathrm{Edeg}x^{(\beta)}\otimes x^{\nu}$. If $\mathrm{Edeg}_i x^{(\alpha)}\otimes x^{\mu}\neq \mathrm{Edeg}_i x^{(\beta)}\otimes x^{\nu}$ for some $i\in I_0$, then $x^{(\alpha)}\otimes x^{\mu}\notin U_q(\mathfrak{gl}(m|n))\cdot x^{(\beta)}\otimes x^{\nu}$, and $x^{(\beta)}\otimes x^{\nu}\notin U_q(\mathfrak{gl}(m|n))\cdot x^{(\alpha)}\otimes x^{\mu}$.

Proposition

Let $x^{(\alpha)}\otimes x^{\mu}$ and $x^{(\beta)}\otimes x^{\nu}\in \Omega_q^{(s)}(m|n,\mathbf{r})$ or $\Omega_q^{(s)}(m|n)$ with $\mathrm{Edeg}_i x^{(\alpha)}\otimes x^{\mu}=\mathrm{Edeg}_i x^{(\beta)}\otimes x^{\nu}$ for each $i\in I_0$.Then there exist $u,v\in U_q(\mathfrak{gl}(m|n))$ such that $u\cdot x^{(\alpha)}\otimes x^{\mu}=x^{(\beta)}\otimes x^{\nu}$, $v\cdot x^{(\beta)}\otimes x^{\nu}=x^{(\alpha)}\otimes x^{\mu}$. In other words, $U_q(\mathfrak{gl}(m|n))\cdot x^{(\alpha)}\otimes x^{\mu}=U_q(\mathfrak{gl}(m|n))\cdot x^{(\beta)}\otimes x^{\nu}$.

Given $0 \le s \le N$, where $N = |\mathbf{r}| + n = m(rl-1)+n$, denoted by $E(s)_0$, $E(s)$ the lowest and highest energy degree of elements in $\Omega_q^{(s)}(m|n,\mathbf{r})$, respectively. The following lemma describes the energy degree of any element in $\Omega_q^{(s)}(m|n,\mathbf{r})$ by considering the different intervals $0 \le s \le N$ falls in.

Lemma

Suppose $m \ge 3$, and $\mathbf{char}(q) = l \ge 3$. For all $0 \le s \le N$, where $N = m(rl-1)+n$.

(1) When $0 \le s \le l-1$: $E(s)_0 = E(s) = 0$.

(2) When $l \le s \le m(l-1)+n$: $E(s)_0 = 0$.

(i) $E(s) = (j - j_1 + n_1') - \sum_{i=-(l-1)}^{-1} \delta_{i,n_2'+h-j_2} + \sum_{i=l}^{2l-2} \delta_{i,n_2'+h-j_2}$, for $s = n' + j(l-1) + h$, where $0 \le n' \le n$, $1 \le j \le m-1$, $0 \le h \le l-1$ and $n' = n_1' l + n_2' (0 \le n_2' \le l-1)$, $j = j_1 l + j_2 (0 \le j_2 \le l-1)$.

(ii) $j - j_1 + n_1' - 1 \le E(s) \le j - j_1 + n_1' + 1$, for $n' + j(l-1) + 1 \le s \le n' + (j+1)(l-1)$ and $1 \le j \le m-1$, $0 \le n' \le n$.

Lemma

Suppose $m \geq 3$, and$\mathbf{char}(q) = l \geq 3$. For any $0 \leq s \leq N$, where $N = m(rl-1)+n$.
(3) When $m(l-1)+n+1 \leq s \leq N-l$:$E(s)_0 = k$, and $k+1 \leq E(s) \leq m(r-1)$, for $s = n+kl+(m-1)(l-1)+h$, where $0 \leq h \leq l-1, 1 \leq k \leq m(r-1)-1$. Namely,
$(\mathrm{i})k + E(n+(m-1)(l-1)) \leq E(s) \leq k + E(n+m(l-1))$, for $k \leq m(r-1) - E(n+(m-1)(l-1))$;
$(\mathrm{ii})E(s) = m(r-1)$, for $k > m(r-1) - E(n+(m-1)(l-1))$, where $E(n+(m-1)(l-1)) \geq 1$ under the assumption $m \geq 3$.
(4) When $N-l+1 \leq s \leq N$: $E(s)_0 = E(s) = m(r-1)$

Theorem

For $U_q(\mathfrak{gl}(m|n))$-module $\Omega_q^{(s)}(m|n, \mathbf{r})$, and $0 \leq s \leq N$, we have
(1) Submodule $\mathfrak{V}_y = U_q(\mathfrak{gl}(m|n)) \cdot y$ is simple if and only if $\mathrm{Edeg}(y) = E(s)_0$, where $s = |y|$.
(2)
Soc $\Omega_q^{(s)}(m|n, \mathbf{r}) = \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)} \otimes x^{\mu} \in \Omega_q^{(s)}(m|n, \mathbf{r}) \mid |\mathcal{E}\langle \alpha, \mu \rangle| = E(s)_0\}$.
(3) $\Omega_q^{(s)}(m|n, \mathbf{r}) = \sum_{\langle \alpha, \mu \rangle \in \mathbb{Z}^m \times \mathbb{Z}_2^n, |\langle \alpha, \mu \rangle| = E(s)} \mathfrak{V}_{\langle \alpha, \mu \rangle}$, where $\mathfrak{V}_{\langle \alpha, \mu \rangle} = U_q(\mathfrak{gl}(m|n)) \cdot x^{(\alpha)} \otimes x^{\mu}$.
(4) When $0 \leq s \leq l-1$, or $N-(l-1) \leq s \leq N$: $\Omega_q^{(s)}(m|n, \mathbf{r}) = \mathfrak{V}_\eta$ is simple.
(5) When $l \leq s \leq N-l$: $\Omega_q^{(s)}(m|n, \mathbf{r}) = \mathfrak{V}_\eta$ is indecomposable.

Definition

Set $\mathcal{V}_0 = \mathrm{Soc}\Omega_q(m|n,\mathbf{r})^{(s)}$, and for $i > 0$,

$$\mathcal{V}_i = \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)}\otimes x^{\mu} \in \Omega_q(m|n,\mathbf{r})^{(s)} \mid E(s)_0 \leq \mathrm{Edeg}x^{(\alpha)}\otimes x^{\mu} \leq E(s)_0+i\}.$$

Theorem

For the indecomposable $U_q(\mathfrak{gl}(m|n))$-module $\Omega_q(m|n,\mathbf{r})$, and $1 \leq s \leq N = m(rl-1)+n$, we have
(1) $\mathcal{V}_i$'s are $U_q(\mathfrak{gl}(m|n))$-submodules of $\Omega_q(m|n,\mathbf{r})$, and the filtration is a Loewy filtration of $\Omega_q^{(s)}(m|n,\mathbf{r})$.

$$0 \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{E(s)-E(s)_0} = \Omega_q^{(s)}(m|n,\mathbf{r}).$$

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Theorem

(2) When $s - |\mathbf{n}|l \leq m(l-1)$, $x^{\eta^1_{(\mathbf{n},j)}}$, where $\eta^1_{(\mathbf{n},j)} \in \chi^1_{i,s}$ are primitive vectors of $\mathcal{V}_i$(relative to $\mathcal{V}_{i-1}$), for all $\mathbf{n} \in \mathcal{N}_i^{(s)}$, and $U_q(\mathfrak{gl}(m|n)) \cdot (x^{(\eta^1_{\mathbf{n},j})} + \mathcal{V}_{i-1}) \cong U_q(\mathfrak{gl}(m|n)) \cdot x^{(\mathbf{s}^1_j)} = \mathfrak{V}_{(\mathbf{s}^1_j)}$. Its i-th Loewy layer

$$
\begin{aligned}
\mathcal{V}_i/\mathcal{V}_{i-1} &= \mathrm{span}_{\mathbb{K}}\{x^{(\alpha)} \otimes x^{\mu} \mid \mathrm{Edeg} x^{(\alpha)} \otimes x^{\mu} = E(s)_0 + i\} \\
&= \bigoplus_{\eta^1_{(\mathbf{n},j)} \in \chi^1_{i,s}} U_q(\mathfrak{gl}(m|n)) \cdot (x^{(\eta^1_{(\mathbf{n},j)})} \otimes 1 + \mathcal{V}_{i-1}) \\
&= (\#\mathcal{N}_i^{(s)}) \mathfrak{V}_{(\mathbf{s}^1_j)}.
\end{aligned}
$$

(3) When $m(l-1) < s - |\mathbf{n}|l \leq m(l-1) + n$, $\cdots$

Loewy filtration of $\Omega_q(m|n, \mathbf{r})$ and its rigidity

A module M is rigid if both the radical filtration and the socle filtration are Loewy filtrations, and $\mathrm{Rad}^{r-k} M \subseteq \mathrm{Soc}^k M$, with $r = \ell\ell M$ is the Loewy length of M.

Theorem

Suppose$m > 2$, and $\mathbf{char} q = l \geq 3$. Then $\Omega_q^{(s)}(m|n, \mathbf{r})$ is a rigid $U_q(\mathfrak{gl}(m|n))$-module, and $\ell\ell\Omega_q^{(s)}(m|n, \mathbf{r}) = E(s) - E(s)_0 + 1$.

Recall the definitions of the quantum Grassmann superalgebra $\Omega_q^{(m|n)}$ and its Manin dual object $\Lambda_q(m|n)$ in [FHZZ,arXiv:1909.10276].

Definition(FHZZ,arXiv:1909.10276)

Denoted by dx_i as the generators of $\Lambda_q(m|n)$, they satisfy

$$\begin{aligned} dx_i^2 = 0, dx_j dx_i = -q^{-1} dx_i dx_j, &\quad \textit{for } i \in I_0, j \in I, j > i; \\ dx_j dx_i = q^{-1} dx_i dx_j, &\quad \textit{for } i, j \in I_1, j > i. \end{aligned}$$

Denote $|\nu| := |dx^\nu| = \sum \nu_i$, $|\beta| := |dx^{(\beta)}| = \sum \beta_i$, let $\Lambda_q(m|n)_{(s)}$ be the s-th homogeneous subspace of $\Lambda_q(m|n)$, that is

$$\Lambda_q(m|n)_{(s)} = \mathrm{span}_{\mathbb{K}}\{dx^\nu \otimes dx^{(\beta)} \mid \nu \in \mathbb{Z}_2^m,\ \beta \in \mathbb{Z}_+^n,\ |\nu| + |\beta| = s\}.$$

Quantum super de Rham cohomology

Definition

The 0-degree q-differential d^0 on $\Omega_q(m|n) \otimes \Lambda_q(m|n)_{(0)}$ is a linear map

$$d^0 : \Omega_q(m|n) \otimes_{\mathbb{K}} \Lambda_q(m|n)_{(0)} \to \Omega_q(m|n) \otimes_{\mathbb{K}} \Lambda_q(m|n)_{(1)}$$

$$d^0(x^{(\alpha)} \otimes x^{\mu}) = \sum_{i=1}^{m} (q^{-\varepsilon_i * \alpha} x^{(\alpha-\varepsilon_i)} \otimes x^{\mu}) \otimes dx_i +$$

$$\sum_{i=m+1}^{m+n-1} (q^{-|\alpha|} (-q)^{-\varepsilon_i * \mu} \delta_{\mu_i,1} x^{(\alpha)} \otimes x^{\mu-\varepsilon_i}) \otimes dx_i$$

In particular, $d^0(1) = 0$ and $d^0(x_i) = 1 \otimes dx_i$.

Proposition

The 0-degree q-differential d^0 is a $U_q(\mathfrak{gl}(m|n))$-module homomorphism of parity 0, that is, $d^0(u \cdot x) = u \cdot d^0(x)$, for $u \in U_q(\mathfrak{gl}(m|n)), x \in \Omega_q(m|n)$.

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Now define the higher degrees q-differentials d^s as the following linear maps

$$d^s : \Omega_q(m|n) \otimes_{\mathbb{K}} \Lambda_q(m|n)_{(s)} \to \Omega_q(m|n) \otimes_{\mathbb{K}} \Lambda_q(m|n)_{(s+1)},$$

$$\begin{aligned} & d^s((x^{(\alpha)} \otimes x^{\mu}) \otimes (dx^{\nu} \otimes dx^{(\beta)})) \\ & = \sum_{j=1}^{m} q^{-\varepsilon_j * \alpha}(x^{(\alpha-\varepsilon_j)} \otimes x^{\mu}) \otimes dx_j \cdot (dx^{\nu} \otimes dx^{(\beta)}) \\ & + \sum_{j=m+1}^{m+n-1} q^{-|\alpha|}(-q)^{-\varepsilon_j * \mu} \delta_{\mu_j,1}(x^{(\alpha)} \otimes x^{\mu-\varepsilon_j}) \otimes dx_j \cdot (dx^{\nu} \otimes dx^{(\beta)}). \end{aligned}$$

Theorem

$(\Omega_q(m|n) \otimes \Lambda_q(m|n), d^{\bullet})$ is a complex, that is, $d^{s+1}d^s = 0$, for $s = 0, 1, 2, \cdots$.

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Definition

$$\mathcal{C}_q(m|n) := \Omega_q(m|n) \otimes \Lambda_q(m|n) = \bigcup_{\mathbf{r}} \mathcal{C}_q(m|n,\mathbf{r}),$$

where,

$$\mathcal{C}_q(m|n,\mathbf{r}) = \mathrm{span}_{\mathbb{K}}\{(x^{(\alpha)} \otimes x^{\mu}) \otimes (dx^{\nu} \otimes dx^{(\beta)}) \in \mathcal{C}_q(m|n) \mid \alpha \leq \mathbf{r}\}.$$

Denote $\mathcal{C}_q(m|n,\mathbf{r})^{(s)} = (\Omega_q(m|n) \times \Lambda_q(m|n))_{(s)} \cap \mathcal{C}_q((m|n),\mathbf{r})$, then

$$\mathcal{C}_q(m|n,\mathbf{r}) = \bigoplus_{s} \mathcal{C}_q(m|n,\mathbf{r})^{(s)}.$$

Theorem

For the quantum subcomplex $(\mathcal{C}_q(m|n,\mathbf{r}), d^\bullet)$ below,

$$0 \to \mathcal{C}_q(m|n,\mathbf{r})^{(0)} \xrightarrow{d^0} \cdots \xrightarrow{d^{s-1}} \mathcal{C}_q(m|n,\mathbf{r})^{(s)} \xrightarrow{d^s} \mathcal{C}_q(m|n,\mathbf{r})^{(s+1)} \xrightarrow{d^{s+1}}$$

$$\cdots \mathcal{C}_q(m|n,\mathbf{r})^{(m+n)} \xrightarrow{d^{m+n}} 0$$

We have

$$\begin{aligned} H^s(\mathcal{C}_q(m|n,\mathbf{r})) &= \mathrm{Ker} d^s / \mathrm{Im} d^{s-1} \\ &\cong \bigoplus_{1\le i_1<i_2\cdots<i_s\le m} \mathbb{K}[x^{(\sum_{j=1}^s (rl-1)\varepsilon_{i_j})} \otimes dx^{\sum_{j=1}^s \varepsilon_{i_j}}], \end{aligned}$$

as $\mathbb{K}$-vector spaces, and $\dim H^s(\mathcal{C}_q(m|n,\mathbf{r}) = \binom{m}{s}$, for $s = 0, 1, \cdots$.

For the quantum subcomplex $(\mathcal{C}_q(m|n,\mathbf{r}), d^{\bullet})$ below,

Definition

Let $V(\epsilon_1,\cdots,\epsilon_{m+n-1})$ be a one-dimentional $U_q(\mathfrak{gl}(m|n))$-module. It is called a **sign-trivial module**, if for any $\mathbf{0}\neq v\in V(\epsilon_1,\cdots,\epsilon_{m+n-1})$, $e_i\cdot v=f_i\cdot v=0$, and $K_i\cdot v=\omega_i v$, where $\omega_i=\pm 1$, for $i=1,\cdots,m+n-1$.

Theorem

For any $s(0\leq s\leq m)$, each cohomology group $H^s(\Omega_q(m|n,\mathbf{r}))$ is isomorphic to the direct sum of $\binom{m}{s}$(sign-trivial)$U_q(\mathfrak{gl}(m|n))$-modules, when q is an l-th(resp.2l-th, but r is odd) root of unit or r is even.

Quantum super de Rham cohomology

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Quantum super de Rham cohomology

Proposition

For the quantum complex $(\mathcal{C}_q(m|n), d^\bullet)$ over $\Omega_q(m|n)$:

$$0 \to \mathcal{C}_q(m|n)^{(0)} \xrightarrow{d^0} \cdots \xrightarrow{d^{s-1}} \mathcal{C}_q(m|n)^{(s)} \xrightarrow{d^s} \Omega_q(m|n)^{(s+1)} \xrightarrow{d^{s+1}}$$

$$\cdots \Omega_q(m|n)^{(m+n)} \xrightarrow{d^{m+n}} 0.$$

We have $H^0(\mathcal{C}_q(m|n)) = \mathbb{K}$, and $H^s(\mathcal{C}_q(m|n)) = 0$, for $0 < s \leq m+n$.

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Thanks for your attention!