Cohen-Macaulay modules of quotient singularities

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2021 Hopf代数会议
8月24日

Based on joint works with Ye Yu and Ma Xinchao

Outline

(I) Motivations

(II) Noncommutative quadric hypersurfaces

(III) Clifford deformations

(IV) Generalized Knörrer's periodicity theorem

(I) Motivations

Basic settings

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  (1) $\mathrm{injdim}_A A = \mathrm{injdim} A_A = d < \infty$
  (2) $\mathrm{Ext}_A^n(\Bbbk, A) = 0$ if $n \neq d$, and $\mathrm{Ext}_A^d(\Bbbk, A) \cong \Bbbk$.

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  If further, $\mathrm{gldim} A = d$, then A is called an Artin-Schelter regular algebra.
- AS-regulars algebras:
  $A = \Bbbk[x_1, \dots, x_n]$.
  $A = \Bbbk_{q_{ij}}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n \rangle / (x_i x_j - q_{ij} x_i x_j : i < j, q_{ij} \neq 0)$.

  AS-Gorenstein algebra: $A = \Bbbk[x, y, z]/(xy - z^2)$.

Finite group (Semisimple Hopf algebra) actions

- $G \leq Aut_{gr} A$ a finite subgroup.

  Question: If A is AS-regular, is it possible that the invariant subalgebra $A^G (:= \{a \in A | g(a) = a, \ \forall g \in G\})$ is still AS-regular?

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- Let A be a connected graded algebra.

  A is called a Koszul algebra if the trivial module $\Bbbk_A$ has a graded free resolution

$$0 \longleftarrow \Bbbk_A \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_n \longleftarrow \cdots$$

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- A quantum polynomial algebra is a Koszul Artin-Schelter regular algebra A such that
  (1) $H_A(t) = (1-t)^{-n}$ for some $n \geq 1$,
  (2) A is a domain.

  $H_A(t) = \sum_{n>0} t^n \dim A_n$.

Shephard-Todd-Chevalley Theorem

Theorem

Let A be a quantum polynomial algebra, and let $G \leq Aut_{gr}(A)$ be a finite subgroup.

(i) Assume A is a polynomial algebra, then A^G is of finite global dimension if and only if G is generated by pseudo-reflections.

(ii) Assume G is an abelian group. Then A^G is AS-regular if and only if G is generated by "noncommutative" pseudo-reflections.

- **G.C. Shephard, J.A. Todd**, Finite unitary reflections groups, Canadian J. Math. 6 (1954), 274–304.

  E. Kirkman, J. Kuzmanovich, J.J. Zhang, Shephard-Todd-Chevalley Theorem for skew polynomial rings, Algebra Repr. Theory 13 (2010), 127–158.

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Theorem

Let A be an AS-regular algebra, and let $G \leq \mathrm{Aut}_{gr}(A)$ be a finite subgroup. If the homological determinant of G is trivial, then A^G is AS-Gorenstein.

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- **Remark.** The above result has been generalized to semisimple Hopf algebra actions by many authors: E. Kirkman, J. Kuzmanovich, Q.-S. Wu, J.J. Zhang, R. Zhu, etc.

Theorem

Let H be a semisimple Hopf algebra, A an AS-regular algebra of global dimension 2. Assume that H acts on A, and the homological determinant is trivial. Then $A^G \cong B/Bf$, for some AS-regular algebra of global dimension 3 and $f \in B$ is a normal regular element.

- **D. Chan, E. Kirkman, C. Walton, J.J. Zhang**, Quantum binary polyhedral groups and their actions on quantum planes, J. reine angew. Math. 719 (2016), 211–252

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- **Example.** $A = \Bbbk[x, y]$, $G = \langle g \rangle$, $g : x \mapsto -x, y \mapsto -y$. Then $A^G \cong \Bbbk[x, y, z]/(xy - z^2)$.

(II) Noncommutative quadric hypersurfaces

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  (1) A_f is a Koszul algebra.

  (2) If A is of global dimension d, then A_f is an Artin-Schelter Gorenstein algebra of injective dimension $d-1$.

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- $A = \Bbbk[x, y]$, $f = x^2 + y^2$, A_f
  $A' = \Bbbk_{-1}[x, y]$, $f = x^2 + y^2$, A'_f

MCM modules

- R a noetherian connected graded algebra.

  $\mathsf{gr}\, R$, category of finitely generated graded right R-modules

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- For $M \in \mathsf{gr}\, R$, let
  $\Gamma(M) = \{m \in M | mA \text{ is finite dimensional}\}$.

  The i-th right derived functor of Γ is denoted by $R^i\Gamma$.

  For $M \in \mathsf{gr}\, R$, the *depth* of M is defined to be the number

$$\mathrm{depth}(M) = \min\{i | R^i\Gamma(M) \neq 0\}.$$

MCM modules

- Suppose that R is an Artin-Schelter Gorenstein algebra with injdimR_R=injdim$_R R = d$.

  $M \in \mathsf{gr}\, R$ is called a maximal Cohen-Macaulay module (MCM module) if depth$(M) = d$.

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- $\mathsf{mcm}\, R$ the category of all the MCM over R.

  $\mathsf{mcm}\, R$ is a Frobenius category, hence the stable category $\underline{\mathsf{mcm}} R$ is a triangulated category.

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- **Example.** $A = \Bbbk[x, y]$, $A' = \Bbbk_{-1}[x, y]$, $f = x^2 + y^2$. $\underline{\operatorname{mcm}} A_f \cong \underline{\operatorname{mcm}} A'_f \cong D^b(\Bbbk \times \Bbbk)$.

- A fundamental result:

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element. Then there is a finite dimensional algebra $C(A_f)$ such that there is an equivalence of triangulated categories

$$D^b(C(A_f)) \cong \underline{\mathrm{mcm}} A_f .$$

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- **Problems:**
  (1) find a way to compute $C(A_f)$;
  (2) Let $A = \Bbbk[x, y]$ and $A' = \Bbbk_{-1}[x, y]$, and let $f = x^2 + y^2$. Note that $C(A_f) \cong C(A'_f) \cong \Bbbk \times \Bbbk$. So, how can we recognize the difference between A_f and A'_f?

(III) Clifford deformations

Clifford deformation of Koszul algebra

- Let V be a finite dimensional vector space, and let $E = T(V)/(R)$ be a Koszul algebra, where $R \subseteq V \otimes V$.

  A linear map $\theta : R \to \Bbbk$ is called a Clifford map if

$$(\theta \otimes 1 - 1 \otimes \theta)(V \otimes R \cap R \otimes V) = 0.$$

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- Note that a Clifford deformation is a special case of Poicaré-Birkhoff-Witt deformations.
- The usual Clifford algebra

  $$\mathbb{R}_n^{p,q} = \mathbb{R}\langle x_1, \ldots, x_n\rangle/(x_i^2+1, x_j^2-1 : 1 \leq i \leq p, p+1 \leq j \leq p+q)$$

  is a Clifford deformation of the exterior algebra $E = \bigwedge\{x_1, \ldots, x_n\}$.

- Let A be a quantum polynomial algebra.

Proposition

Let $E = A^!$ be the quadratic dual of the quantum polynomial algebra A. Then E is a Koszul Frobenius algebra.

S.P. Smith, Some finite dimensional algebras related to elliptic curves, in: CMS Conf. Proc., 1996

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- We have the following facts:

Proposition

- *Each central element $0 \neq f \in A_2$ is corresponding to a Clifford map θ_f of $E = A^!$.*
- *The Clifford deformation $E(\theta_f)$ is a strongly $\mathbb{Z}_2$-graded algebra.*
- $C(A_f) \cong E(\theta_f)_0$.

Example

- Let $A = \Bbbk\langle x, y, z\rangle/(r_1, r_2, r_3)$, where $r_1 = zx + xz$, $r_2 = yz + zy$, $r_3 = x^2 + y^2$. Then A is a quantum polynomial algebra of dimension 3.

f	$C(A_f) = E(\theta_f)_0$
$z^2 + xy + yx + \lambda x^2$	$\mathbf{k}^4$
$z^2 + xy + yx \pm 2\sqrt{-1}x^2$	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$
z^2	$\mathbf{k}[u, v]/(u^2 - v^2, uv)$
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x^2	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$

- **Remark.** Let A be a quantum polynomial algebra of dimensional 3. If $f \in A_2$ is a central element, $A_f = A/Af$ is called a noncommutative conic. The algebras $C(A_f)$ have been classified for noncommutative conics.

H. Hu, Classification of noncommutative conics associated to symmetric regular superpotentials, arXiv:2005.03918.

H. Hu, M. Matsuno, I. Mori, Noncommutative conics in Calabi-Yau quantum planes,

arXiv:2104.00221.

Theorem

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central regular element.

Then $\operatorname{qgr} A_f$ has finite global dimension (i.e., $\operatorname{proj} A_f$ is smooth) if and only if $C(A_f) = E(\theta_f)_0$ is a semisimple algebra.

S. P. Smith, M. Van den Bergh, Noncommutative quadric surfaces, J. Noncommut. Geom. 7 (2013), 817–856.

J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, arxiv:1905.04699

I. Mori, K. Ueyama, Noncommutative Knörrer Periodicity Theorem and noncommutative quadric hypersurfaces, arxiv:1905.12266

(IV) Generalizations of Knörrer's periodicity theorem

An example

- **Example.** $A = \Bbbk[x, y]$, $A' = \Bbbk_{-1}[x, y]$, $f = x^2 + y^2$.

  Then $\underline{\mathsf{mcm}}(A_f) \cong \underline{\mathsf{mcm}}(A'_f) \cong D^b(\Bbbk \times \Bbbk)$.

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- Let B be a quantum polynomial algebra and $g \in B_2$ be a central element.

  Consider the tensor algebra $B \otimes A$ and $B \otimes A'$, and view $h := g + f$ as an element in $B \otimes A$ (or in $B \otimes A'$, resp.).

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- Fact: $\underline{\mathsf{mcm}}(B \otimes A)_h$ is different from $\underline{\mathsf{mcm}}(B \otimes A')_h$!

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- Fact: $\underline{\mathrm{mcm}}(B \otimes A)_h$ is different from $\underline{\mathrm{mcm}}(B \otimes A')_h$!

- The reason is the following:

  Let E and E' be the Koszul dual of A and A' respectively.

  The Clifford deformation of E and that of E' associated to f are very different!

  Indeed, $E(\theta_f) \cong \mathbb{M}_2(\Bbbk)$ and $E'(\theta_f) \cong \Bbbk\mathbb{Z}_2 \times \Bbbk\mathbb{Z}_2$.

Definition

- It seems reasonable to classify the noncommutative quadric hypersurfaces according the Clifford deformations.

Definition

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central element. Let E be the Koszul dual of A.

If the Clifford deformation $E(\theta_f)$ is a simple $\mathbb{Z}_2$-graded algebra, then we call $A_f = A/Af$ is a simple graded isolated singularity.

Definition

- Since $\Bbbk$ is algebraically closed, there are only two classes of simple $\mathbb{Z}_2$-graded algebra:

  (0) matrix algebras over $\Bbbk$;

  (1) matrix algebras over $\Bbbk\mathbb{Z}_2$.

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- If $E(\theta_f)$ is a matrix algebra over $\Bbbk$, then we further call A_f is a simple graded isolated singularity of 0-type

  If $E(\theta_f)$ is a matrix algebra over $\Bbbk\mathbb{Z}_2$, then we further call A_f is a simple graded isolated singularity of 1-type.

- **Proposition.** Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f + g \in A \otimes B$.
    - If both A_f and B_f are simple graded isolated singularity of 1-type, then $(A \otimes B)_h$ is a simple graded isolated singularity of 0-type.
    - If A_f is a simple graded isolated singularity of 1-type and B is a simple graded isolated singularity of 0-type, then$(A \otimes B)_h$ is a simple graded isolated singularity of 1-type.
    - If both A_f and B_f are simple graded isolated singularity of 0-type, then $(A \otimes B)_h$ is a simple graded isolated singularity of 0-type.

- A key lemma.

Lemma

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f + g \in A \otimes B$.

Then we have an isomorphism of $\mathbb{Z}_2$-graded algebras

$$E_{(A\otimes B)^!}(\theta_h) \cong E_{A^!}(\theta_f)\hat{\otimes}E_{B^!}(\theta_g),$$

where $\hat{\otimes}$ is the $\mathbb{Z}_2$-graded tensor.

- Let $A = \mathbb{k}[x, y]$, and $f = x^2 + y^2$. Then

$$E(\theta_f) \cong \mathbb{M}_2(\mathbb{k}),$$

where $\mathbb{M}_2(\mathbb{k})$ is viewed as a $\mathbb{Z}_2$-graded algebra by setting

$$\mathbb{M}_2(\mathbb{k})_0 = \begin{bmatrix} \mathbb{k} & 0 \\ 0 & \mathbb{k} \end{bmatrix}, \quad \mathbb{M}_2(\mathbb{k})_1 = \begin{bmatrix} 0 & \mathbb{k} \\ \mathbb{k} & 0 \end{bmatrix}.$$

Hence A_f is a simple graded isolated singularity of 0-type.

Examples

- Let $A = \Bbbk[x, y]$, and $f = x^2 + y^2$. Then
$$E(\theta_f) \cong \mathbb{M}_2(\Bbbk),$$
where $\mathbb{M}_2(\Bbbk)$ is viewed as a $\mathbb{Z}_2$-graded algebra by setting
$$\mathbb{M}_2(\Bbbk)_0 = \begin{bmatrix} \Bbbk & 0 \\ 0 & \Bbbk \end{bmatrix}, \quad \mathbb{M}_2(\Bbbk)_1 = \begin{bmatrix} 0 & \Bbbk \\ \Bbbk & 0 \end{bmatrix}.$$
Hence A_f is a simple graded isolated singularity of 0-type.
- Let $A = \Bbbk\langle x_1, \ldots, x_5\rangle/(r_1, \ldots, r_{10})$, where the generating relations are as follows:
$$r_1 = x_1x_2 - x_2x_1, r_2 = x_1x_3 + x_3x_1, r_3 = x_1x_4 + x_4x_1,$$
$$r_4 = x_1x_5 + x_5x_1, r_5 = x_2x_3 - x_3x_2, r_6 = x_2x_4 + x_4x_1,$$
$$r_7 = x_2x_5 + x_5x_2, r_8 = x_3x_4 - x_4x_3, r_9 = x_3x_5 + x_5x_3,$$
$$r_{10} = x_4x_5 + x_5x_4.$$
Let $f = x_1^2 + \cdots + x_5^2$. Then A_f is a simple graded isolated singularity of 1-type.

- **Remark.** We are unable to find a way to characterize when A_f is a simple graded isolated singularity.

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- **Proposition.** Let A be a quantum polynomial algebra of global dimension n, and let $f \in A_2$ be a central regular element.
  (i) If A_f is a simple graded isolated singularity of 0-type, then n is even.
  (ii) If A_f is a simple graded isolated singularity of 1-type, then n is odd.

- We have the following generalized Knörrer's periodicity theorem.

Theorem

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian, and let $h = f + g \in A \otimes B$.

(i) *If B_g is a simple graded isolated singularity of 0-tpye, then there are equivalences of triangulated categories*
$$\underline{\mathrm{mcm}}(A \otimes B)_h \cong D^b(\mathrm{mod}\,E(\theta_f)_0) \cong \underline{\mathrm{mcm}}A_f;$$

(ii) *If B_g is a simple graded isolated singularity of 1-type, there is an equivalence of triangulated categories*
$$\underline{\mathrm{mcm}}(A \otimes B)_h \cong D^b(\mathrm{mod}\,E(\theta_f)^\natural),$$
where $E(\theta_f)^\natural$ is the underlying ungraded algebra, and $\mathrm{mod}\,E(\theta_f)^\natural$ is the category of all the finite dimensional modules over $E(\theta_f)^\natural$.

J.-W. He, X.-C. Ma, Y. Ye, Generalized Knörrer Periodicity Theorem, preprint, 2021.

Noncommutative Knörrer's periodicity theorem

- In particular, if we take $B = \Bbbk[x, y]$ and $g = x^2 + y^2$.

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element.
(i) Let $A_f^{\#\#} = A[x, y]/(f + x^2 + y^2)$. Then

$$\underline{\mathrm{mcm}} A_f^{\#\#} \cong \underline{\mathrm{mcm}} A_f \cong D^b(\mathrm{mod}\, E(\theta_f)_0).$$

(ii)Let $A_f^{\#} = A[x]/(f + x^2)$. Then $\underline{\mathrm{mcm}} A_f^{\#} \cong D^b(\mathrm{mod}\, E(\theta_f)^{\natural})$.

H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, Invent. Math. 88 (1987), 153–164.

A. Conner, E. Kirkman, W. F. Moore, C. Walton, Noncommutative Knörrer periodicity and noncommutative Kleinian singularities, arXiv:1809.06524.

J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, arxiv:1905.04699.

I. Mori, K. Ueyama, Noncommutative Knörrer's Periodicity Theorem and noncommutative quadric surfaces, arXiv:1905.12266.

An example

- Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central element. Taking a minimal graded projective resolution of $\Bbbk_{A_f}$ as follows:

$$\cdots \longrightarrow P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \xrightarrow{\partial^{-d+1}} \cdots \longrightarrow P^0 \longrightarrow \Bbbk_{A_f} \longrightarrow 0.$$

Let $\Omega^d(\Bbbk_{A_f}) = \ker \partial^{-d+1}$ be the dth syzygy of the trivial module.

We fix a notion as follows:

$$\mathbb{M} := \Omega^d(\Bbbk_{A_f})(d).$$

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Theorem

We have $\mathrm{End}_{\mathrm{gr}\,A_f}(\mathbb{M}) \cong E(\theta_f)_0 \cong C(A_f)$.

J.-W. He, Y. Ye, Pre-resolutions of noncommutative isolated singularities, arXiv:2005.11873.

An example

- Let $\Bbbk = \mathbb{C}$, let $S = \Bbbk\langle x, y, z\rangle/(R)$, where $R = \mathrm{span}\{xz + zx, yz + zy, x^2 + y^2\}$.

  Let $f = x^2 + z^2 \in S_2$, and $A = S/Sf$.

  Then $\mathrm{injdim} A_A = 2$.

An example

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  Let $f = x^2 + z^2 \in S_2$, and $A = S/Sf$.

  Then $\mathrm{injdim} A_A = 2$.

- Set $\mathbb{M} = \Omega^2(\Bbbk_A)(2)$.

  $\mathbb{M}$ is a quotient module of the free module $m_1A \oplus m_2A \oplus m_3A \oplus m_4A$ with relations:

$$
\begin{aligned}
r_1 &= m_1x + m_2y + m_3z, \\
r_2 &= 2m_1x + m_2y + m_3z + m_4z, \\
r_3 &= m_2z - m_3y + m_4y, \\
r_4 &= m_1z + m_2z - m_3y + m_4(x + y).
\end{aligned}
$$

An example

- $$\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M}) = \left\{ \begin{pmatrix} b+d & 0 & a & a \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ a & c & d & b+d \end{pmatrix} : a, b, c, d \in \mathbb{k} \right\}.$$

- $$\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M}) = \left\{ \begin{pmatrix} b+d & 0 & a & a \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ a & c & d & b+d \end{pmatrix} : a, b, c, d \in \mathbb{k} \right\}.$$
- We have the following complete set of primitive idempotents in $\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M})$:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad e_4 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

An example

- We have the following nonprojective nonisomorphic indecomposable MCM modules:

$$\mathbb{M}^1 := A/(y+iz)A,\ \mathbb{M}^2 := A/(y-iz)A,$$

$$\mathbb{M}^3 := A/(x+z)A,\ \mathbb{M}^4 := A/(x-z)A.$$

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