Deformations of the restricted quantum group $\overline{U}_q(sl_2^*)$ and preprojective algebras

Joint work with Yongjun Xu

Jialei Chen

Beijing University of Technology

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Outline

1. Some notations
2. The restricted quantum group $\overline{U}_q(sl_2^*)$ and its Hopf PBW-deformations
3. Realization of the Hopf algebras $\overline{U}_q(sl_2^*, \kappa)$ via (deformed) preprojective algebra corresponding to Gabriel quiver
4. The category of finite dimensional representations of $\overline{U}_q(sl_2^*)$

In this talk, we works over the complex field $\mathbb{C}$. Fix an integer $n \geq 3$ $(n \neq 4)$. we always assume that q is a primitive n-th root of unity, and

$$d = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

For a invertible element $v \in \mathbb{C}$, and any integer $l > 0$, set

$$(l)_v = 1 + v + \cdots + v^{l-1} = \frac{v^l - 1}{v - 1}.$$

Define the v-factorial of l by $(0)!_v = 1$ and for $l > 0$

$$(l)!_v = (1)_v (2)_v \cdots (l)_v.$$

$$\left(\begin{array}{c} k \\ i \end{array} \right)_v = \frac{(k)!_v}{(i)!_v \cdot (k-i)!_v}$$

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Definition

The restricted quantum algebra $\overline{U}_q(sl_2^*)$ is the associative unital algebra generated by K, K^{-1}, E, F and subject to the following relations

$$KK^{-1} = K^{-1}K = 1, \quad K^d = 1, \quad E^d = F^d = 0,$$
$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF = FE.$$

Remark

- The set $\{F^iK^kE^j | i,j,k \in \mathbb{Z}, 0 \leq i,j,k < d\}$ is a basis of $\overline{U}_q(sl_2^*)$, and the dimension of $\overline{U}_q(sl_2^*)$ is equal to d^3.
- As a $\mathbb{C}$-algebra, $\overline{U}_q(sl_2^*)$ is isomorphic to the smash product algebra $\overline{A}\sharp\mathbb{C}\overline{G}$, i.e., $\overline{U}_q(sl_2^*) \cong \overline{A}\sharp\mathbb{C}\overline{G}$, where the algebra $\overline{A}$ and the abelian group $\overline{G}$ are defined as follows

$$\begin{aligned} \overline{A} &= \mathbb{C}\langle E, F | EF = FE, E^d = F^d = 0\rangle, \\ \overline{G} &= \langle K | K^d = 1\rangle, \end{aligned}$$

and the action of $\overline{G}$ on $\overline{A}$ is given by $K \circ E = q^2E$ and $K \circ F = q^{-2}F$.

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Proposition

$\overline{U}_q(sl_2^*)$ is a Hopf algebra with coproduct Δ, counit ε and antipode S defined by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

Proposition

$\overline{U}_q(sl_2^*)$ is pointed, basic, and nonsemisimple.

Sketch of proof. Let $\mathrm{rad}\big(\overline{U}_q(sl_2^*)\big)$ be the radical of $\overline{U}_q(sl_2^*)$. Denote by $\langle E, F\rangle$ the ideal of $\overline{U}_q(sl_2^*)$ generated by E and F. Note that the ideal $\langle E, F\rangle$ is nilpotent and

$$\overline{U}_q(sl_2^*)/\langle E, F\rangle = \mathbb{C}\langle K | K^d = 1\rangle \cong \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C},$$

therefore $\overline{U}_q(sl_2^*)$ is a basic Hopf algebra. We also get that $\mathrm{rad}\big(\overline{U}_q(sl_2^*)\big) = \langle E, F\rangle$, and then the nonsemisimplicity of $\overline{U}_q(sl_2^*)$ is obtained. Pointed is clearly.

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Definition

The quantum algebra $\overline{U}_q(sl_2^*, \kappa)$ is the associative $\mathbb{C}$-algebra with unit 1 generated by E, F, K, and K^{-1}, subject to the following relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad KK^{-1} = K^{-1}K = 1,$$

$$K^d = 1 \quad , E^d = F^d = 0, \quad EF - FE = a(K^m - K^{-m}),$$

where $m \in \mathbb{Z}$ and $1 \leq m < d$.

Definition

We call $\overline{U}_q(sl_2^*, \kappa)$ a Hopf PBW-deformation of $\overline{U}_q(sl_2^*)$, if $\overline{U}_q(sl_2^*, \kappa)$ is a PBW-deformation of $\overline{U}_q(sl_2^*)$, and has the Hopf algebra structure as follows

$$\begin{aligned}
&\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\
&\Delta(E) = E \otimes K^t + K^s \otimes E, \quad \Delta(F) = F \otimes K^{-s} + K^{-t} \otimes F, \\
&\varepsilon(K) = \varepsilon(K^{-1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0, \\
&S(K) = K^{-1}, \quad S(K^{-1}) = K, \\
&S(E) = -K^{-s} E K^{-t}, \quad S(F) = -K^t F K^s,
\end{aligned} \tag{1}$$

where $s, t \in \mathbb{Z}$ with $t - s = m$.

Remark

For a given m, all the $\overline{U}_q(sl_2^*, \kappa)$ with $t - s = m$ are isomorphic.

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For a given m, all the $\overline{U}_q(sl_2^*, \kappa)$ with $t - s = m$ are isomorphic.

Theorem

The quantum algebra $\overline{U}_q(sl_2^*, \kappa)$ *is a Hopf PBW-deformation of* $\overline{U}_q(sl_2^*)$ *if and only if* $(2m, n) = 1$ *when* n *is odd, while* $(m, \frac{n}{2}) = 1$ *when* n *is even.*

Lemma

Assume that n *is odd (resp. even) and* $1 \leq m < d$. *Then* $\begin{pmatrix} d \\ i \end{pmatrix}_{q^{2m}} = 0$ *for* $0 < i < d$ *if and only if* $(2m, d) = 1$ *(resp.* $(m, d) = 1$*).*

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Proof. Firstly, $U_q(sl_2^*, \kappa)$ is a PBW-deformation of $U_q(sl_2^*)$ when q is a primitive n-th root of unity with $n \geq 3 (n \neq 4)$. Note that

$$\overline{U}_q(sl_2^*) \cong U_q(sl_2^*)/\langle K^d - 1, E^d, F^d \rangle,$$
$$\overline{U}_q(sl_2^*, \kappa) \cong U_q(sl_2^*, \kappa)/\langle K^d - 1, E^d, F^d \rangle,$$

then $\overline{U}_q(sl_2^*, \kappa)$ is a PBW-deformation of $\overline{U}_q(sl_2^*)$ as $\langle K^d - 1, E^d, F^d \rangle$ is a homogenous ideal.
Secondly, $\overline{U}_q(sl_2^*, \kappa)$ is a Hopf algebra with the structure maps in (1) if and only if $(2m, n) = 1$ when n is odd, while $(m, \frac{n}{2}) = 1$ when n is even.

For $0 \leq j \leq d-1$, set

$$\epsilon_j = \frac{1}{d}\sum_{i=0}^{d-1} q^{2ij} K^i,$$

then

$$\sum_{i=0}^{d-1} \epsilon_i = \frac{1}{d}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} q^{2ij} K^j = \frac{1}{d}\sum_{j=0}^{d-1}\left(\sum_{i=0}^{d-1}\left(q^{2j}\right)^i\right) K^j = 1,$$

$$\epsilon_j \epsilon_l = \frac{1}{d}\sum_{s=0}^{d-1} q^{2ls}\epsilon_j K^s = \frac{1}{d}\sum_{s=0}^{d-1}\left(q^{2(l-j)}\right)^s \epsilon_j = \begin{cases} \epsilon_j, & \text{if } l = j, \\ 0, & \text{if } l \neq j. \end{cases}$$

Lemma

$\{\epsilon_0, \cdots, \epsilon_{d-1}\}$ is a complete set of primitive orthogonal idempotents of $\overline{U}_q(sl_2^*)$.

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Lemma

$\{\epsilon_0, \cdots, \epsilon_{d-1}\}$ *is a complete set of primitive orthogonal idempotents of* $\overline{U}_q(sl_2^*)$.

We draw the Gabriel quiver $\Gamma = (\Gamma_0, \Gamma_1)$ corresponding to $\overline{U}_q(sl_2^*)$

(1) There are d vertexes $s_0, s_1, \cdots, s_{d-1}$ in Γ_0 which are in correspondence with the idempotents $\epsilon_0, \epsilon_1, \cdots, \epsilon_{d-1}$;

(2) As linear spaces,

$$\mathrm{rad}(\overline{U}_q(sl_2^*)) = \bigoplus_{\substack{1 \le i < d \\ 0 \le k < d-1}} \mathbb{C}F^i K^k \oplus \bigoplus_{\substack{1 \le i < d \\ 0 \le k < d-1}} \mathbb{C}E^i K^k \oplus \bigoplus_{\substack{1 \le r,s < d \\ 0 \le k < d-1}} \mathbb{C}F^r K^k E^s,$$

$$\mathrm{rad}^2(\overline{U}_q(sl_2^*)) = \bigoplus_{\substack{2 \le i < d \\ 0 \le k < d-1}} \mathbb{C}F^i K^k \oplus \bigoplus_{\substack{2 \le i < d \\ 0 \le k < d-1}} \mathbb{C}E^i K^k \oplus \bigoplus_{\substack{1 \le r,s < d \\ 0 \le k < d-1}} \mathbb{C}F^r K^k E^s.$$

$$\dim\left[\mathrm{rad}(\overline{U}_q(sl_2^*))\right] = d^3 - d, \quad \dim\left[\mathrm{rad}^2(\overline{U}_q(sl_2^*))\right] = d^3 - 3d.$$

We draw the Gabriel quiver $\Gamma = (\Gamma_0, \Gamma_1)$ corresponding to $\overline{U}_q(sl_2^*)$
(1) There are d vertexes $s_0, s_1, \cdots, s_{d-1}$ in Γ_0 which are in correspondence with the idempotents $\epsilon_0, \epsilon_1, \cdots, \epsilon_{d-1}$;
(2) As linear spaces,

$$\mathrm{rad}\big(\overline{U}_q(sl_2^*)\big) = \bigoplus_{\substack{1\le i<d\\ 0\le k<d-1}} \mathbb{C}F^iK^k \oplus \bigoplus_{\substack{1\le i<d\\ 0\le k<d-1}} \mathbb{C}E^iK^k \oplus \bigoplus_{\substack{1\le r,s<d\\ 0\le k<d-1}} \mathbb{C}F^rK^kE^s,$$

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Therefore,

$$\mathrm{rad}\big(\overline{U}_q(sl_2^*)\big)/\mathrm{rad}^2\big(\overline{U}_q(sl_2^*)\big) = \bigoplus_{0\leq k<d-1} \mathbb{C}FK^k \oplus \bigoplus_{0\leq k<d-1} \mathbb{C}EK^k.$$

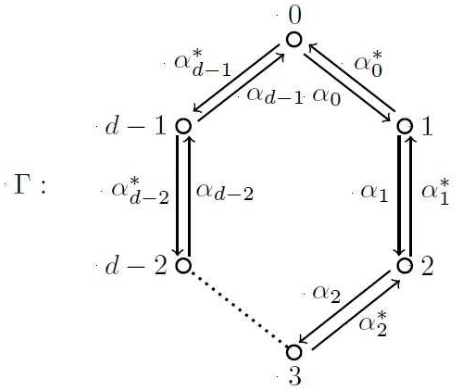
For any two vertexes $a, b \in \Gamma_0$, note that

$$K^i\epsilon_a = q^{-2ai}\epsilon_a, \;\; K^i = \sum_{j\in\mathbb{Z}_d} q^{-2ij}\epsilon_j, \;\; E\epsilon_a = \epsilon_{a-1}E, \;\; F\epsilon_a = \epsilon_{a+1}F,$$

we have

$$\epsilon_a\Big(\mathrm{rad}\big(\overline{U}_q(sl_2^*)\big)/\mathrm{rad}^2\big(\overline{U}_q(sl_2^*)\big)\Big)\epsilon_b = \begin{cases} \mathbb{C}E\epsilon_b, & \text{if } b = a+1, \\ \mathbb{C}F\epsilon_b, & \text{if } a = b+1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we obtain the Gabriel quiver Γ as follows:



which is just the double quiver of affine Dynkin type $\widetilde{\mathbb{A}}_{d-1}$.

Definition

Assume that $a \in \mathbb{C}$ and $1 \leq m < d$ satisfying

$$\begin{cases} m = 1, & \text{when } a = 0, \\ (2m, d) = 1, & \text{when } a \neq 0 \text{ and } n \text{ is odd}, \\ (m, d) = 1, & \text{when } a \neq 0 \text{ and } n \text{ is even}. \end{cases}$$

We define $\Pi_a^m(\Gamma)$ to be the following quotient algebra of path algebra $\mathbb{C}\Gamma$:

$$\Pi_a^m(\Gamma) = \frac{\mathbb{C}\Gamma}{\left\langle \sum_{i \in \mathbb{Z}_d} \left(\alpha_i^* \alpha_i - \alpha_{i-1} \alpha_{i-1}^* - a(q^{-2mi} - q^{2mi}) s_i \right) \right\rangle}.$$

Remark

$\Pi_0^1(\Gamma)$ is just the preprojective algebra corresponding to Γ, while $\Pi_a^m(\Gamma)(a \neq 0)$ is a deformed preprojective algebra.

Remark

In $\mathbb{C}\Gamma$ or $\Pi_a^m(\Gamma)$, the paths are from right to left: for paths α and β, which starts at s_i (resp. s_j), and ends at s_j (resp. s_k), then the multiplication of α and β in $\mathbb{C}\Gamma$ or $\Pi_a^m(\Gamma)$ is denoted by

$$\alpha * \beta = \beta\alpha.$$

In the following proposition we will prove that there are some natural Hopf algebra structures on $\Pi_a^m(\Gamma)$.

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Proposition

The algebra $\Pi_a^m(\Gamma)$ is a Hopf alagebra with comultiplication Δ, counit ε and antipode S as follows:

$$\begin{aligned}
&\Delta(s_l)=\sum_{i+j=l} s_i\otimes s_j,\quad \varepsilon(s_l)=\delta_{l,0},\quad S(s_l)=s_{-l},\\
&\Delta(\alpha_l)=\sum_{i+j=l} q^{-2si}s_i\otimes\alpha_j+\sum_{i+j=l} q^{-2tj}\alpha_i\otimes s_j,\\
&S(\alpha_l)=-q^{-2l(t+s)-2s}\alpha_{-l-1},\quad \varepsilon(\alpha_l)=0,\\
&\Delta(\alpha_l^*)=\sum_{i+j=l} q^{2ti}s_i\otimes\alpha_j^*+\sum_{i+j=l} q^{2sj}\alpha_i^*\otimes s_j,\\
&S(\alpha_l^*)=-q^{2l(t+s)+2s}\alpha_{-l-1}^*,\quad \varepsilon(\alpha_l^*)=0.
\end{aligned}\tag{2}$$

where $i,j,l\in\mathbb{Z}_d$, $s,t\in\mathbb{Z}$ and $t-s=m$.

Proof. (1) We firstly show that the formulas in (2) can induce the following algebra homomorphisms

$$\begin{cases} \Delta : \Pi_a^m(\Gamma) \longrightarrow \Pi_a^m(\Gamma) \otimes \Pi_a^m(\Gamma) \\ \varepsilon : \Pi_a^m(\Gamma) \longrightarrow \mathbb{C}, \\ S : \Pi_a^m(\Gamma) \longrightarrow [\Pi_a^m(\Gamma)]^{op}. \end{cases}$$

By the universal property of path algebra and fundamental homomorphism theorem of algebras, we only need to check that

$$\begin{cases} \sum_{l\in\mathbb{Z}_d} \phi(s_l) = 1, \\ \phi(s_l)^2 = \phi(s_l), \\ \phi(s_k)\phi(s_l) = 0 \text{ for } k \neq l, \\ \phi(\alpha_l) = \phi(s_{l+1})\phi(\alpha_l)\phi(s_l), \\ \phi(\alpha_l^*) = \phi(s_l)\phi(\alpha_l^*)\phi(s_{l+1}), \\ \phi(\alpha_l^*)\phi(\alpha_l) - \phi(\alpha_{l-1})\phi(\alpha_{l-1}^*) = a(q^{-2ml} - q^{2ml})\phi(s_l), \end{cases} \tag{3}$$

where $\phi = \Delta$ (resp. $\phi = \varepsilon$) and 1 is the identity element in $\Pi_a^m(\Gamma) \otimes \Pi_a^m(\Gamma)$ (resp. $\mathbb{C}$), and

$$\begin{cases} \sum\limits_{l\in\mathbb{Z}_d} S(s_l) = 1, \\ S(s_l)^2 = S(s_l), \\ S(s_k)S(s_l) = 0 \text{ for } k \neq l, \\ S(\alpha_l) = S(s_l)S(\alpha_l)S(s_{l+1}), \\ S(\alpha_l^*) = S(s_{l+1})S(\alpha_l^*)S(s_l), \\ S(\alpha_l)S(\alpha_l^*) - S(\alpha_{l-1}^*)S(\alpha_{l-1}) = a(q^{-2ml} - q^{2ml})S(s_l), \end{cases} \quad (4)$$

where 1 is the identity element in $\Pi_a^m(\Gamma)$.

(2) By part (1), to prove $(\Pi_a^m(\Gamma), \Delta, \varepsilon, S)$ is a Hopf algebra, we only need to check that

$$\begin{cases} (\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x), \\ (\varepsilon \otimes id)\Delta(x) = id = (id \otimes \varepsilon)\Delta(x), \\ (S \otimes id)\Delta(x) = \varepsilon(x)1 = (id \otimes S)\Delta(x) \end{cases}$$

for any $x \in \Gamma_0 \cup \Gamma_1 = \{s_l, \alpha_l, \alpha_l^* | l \in \mathbb{Z}_d\}$.

For any integer $l > 0$, define (q,t,s)-number

$$(l)_{q,t,s} = \frac{q^{tl} - q^{sl}}{q^t - q^s}. \tag{5}$$

Define the (q,t,s)-factorial of l by $(0)!_{q,t,s} = 1$ and for $l > 0$

$$(l)!_{q,t,s} = (1)_{q,t,s}(2)_{q,t,s} \cdots (l)_{q,t,s}. \tag{6}$$

We define the (q,t,s)-Gauss polynomials for $0 \leq k \leq l$ by

$$\left(\begin{array}{c} l \\ k \end{array} \right)_{q,t,s} = \frac{(l)!_{q,t,s}}{(k)!_{q,t,s}(l-k)!_{q,t,s}}. \tag{7}$$

Proposition

Let $0 \leq k \leq l$.

(1)

$$\binom{l}{k}_{q,1,0} = \binom{l}{k}_q, \quad \binom{l}{k}_{q,t,s} = \binom{l}{l-k}_{q,t,s}.$$

(2) ((q,t,s)-Pascal identity)

$$\begin{aligned} \binom{l}{k}_{q,t,s} &= q^{s(l-k)} \binom{l-1}{k-1}_{q,t,s} + q^{tk} \binom{l-1}{k}_{q,t,s} \\ &= q^{sk} \binom{l-1}{k}_{q,t,s} + q^{t(l-k)} \binom{l-1}{k-1}_{q,t,s}. \end{aligned}$$

Proof.

$$\begin{cases} (l)_{q,t,s} = q^{s(l-1)}(l)_{q^m}, \\ (l)!_{q,t,s} = q^{\frac{sl(l-1)}{2}}(l)!_{q^m}, \\ \left(\begin{array}{c} l \\ k \end{array}\right)_{q,t,s} = q^{sk(l-k)}\left(\begin{array}{c} l \\ k \end{array}\right)_{q^m}, \end{cases} \tag{8}$$

we can obtain all the above results by using Proposition IV.2.1 in [Kassel, C. Quantum Groups, Graduate Texts in Mathematics; Springer-Verlag, 1995; Vol. 155.] □

Lemma

For $l \geq 0$, let $\gamma_i^l = \alpha_{i+l-1} \cdots \alpha_{i+1}\alpha_i$, which starts at the vertex s_i and has length l, and $\gamma_i^0 = s_i$. Let $(\gamma_i^l)^* = \alpha_i^* \alpha_{i+1}^* \cdots \alpha_{i+l-1}^*$. Then we have

$$\Delta(\gamma_i^l) = \sum_{j+k=i, u+v=l} \begin{pmatrix} l \\ v \end{pmatrix}_{q^{-2},t,s} q^{-2tku-2sjv} \gamma_j^u \otimes \gamma_k^v,$$

$$\Delta((\gamma_i^l)^*) = \sum_{j+k=i, u+v=l} \begin{pmatrix} l \\ v \end{pmatrix}_{q^{2},t,s} q^{2tjv+2sku} (\gamma_j^u)^* \otimes (\gamma_k^v)^*.$$

Lemma

In $\Pi_0^1(\Gamma)$, for any integer $i, j \in \mathbb{Z}_d$, we have

$$\alpha^*_{j+v-1}\gamma^v_j = \gamma^v_{j-1}\alpha^*_{j-1}, \tag{9}$$

$$(\gamma^u_i)^*\gamma^v_j = \gamma^v_{j-u}(\gamma^u_{i-v})^*, \tag{10}$$

where $u + i = v + j$.

Proposition

Let $I_d =: \langle \gamma_i^d, (\gamma_i^d)^* | i \in \mathbb{Z}_d \rangle$ be the ideal of $\Pi_a^m(\Gamma)$, and $\Pi_a^m(\Gamma, I_d)$ be the quotient of (deformed) preprojective algebra $\Pi_a^m(\Gamma)$ module I_d, i.e.,

$$\Pi_a^m(\Gamma, I_d) := \Pi_a^m(\Gamma)/I_d.$$

Then the following statements hold.
(1) $(\Pi_a^m(\Gamma, I_d), \Delta, \varepsilon, S)$ is a Hopf algebra with Δ, ε and S defined in (2).
(2) $\Pi_a^m(\Gamma, I_d)$ has a basis $\{\gamma_i^u(\gamma_i^v)^* | u, v, i \in \mathbb{Z}_d\}$ and

$$\dim \Pi_a^m(\Gamma, I_d) = d^3.$$

Sketch of proof. (1) We prove I_d is a Hopf ideal of $\Pi_a^m(\Gamma)$.

$$\Delta(\gamma_i^d) = \sum_{j+k=i} \left(\gamma_j^0 \otimes \gamma_k^d + \gamma_j^d \otimes \gamma_k^0\right),$$

$$\Delta((\gamma_i^d)^*) = \sum_{j+k=i} \left[(\gamma_j^0)^* \otimes \gamma_k^d + (\gamma_j^d)^* \otimes (\gamma_k^0)^*\right],$$

which implies $\Delta(I_d) \subseteq \Pi_a^m(\Gamma) \otimes I_d + I_d \otimes \Pi_a^m(\Gamma)$. Obviously we have $\varepsilon(\gamma_i^d) = 0$ and $\varepsilon((\gamma_i^d)^*) = 0$. Hence $\varepsilon(I_d) = 0$. Moreover, since

$$\begin{aligned} S(\gamma_i^d) &= S(\alpha_i)S(\alpha_{i+1})\cdots S(\alpha_{i+d-2})S(\alpha_{i+d-1}) \\ &= (-1)^d q^{-d(d-1)(t+s)}\gamma_{-i-d}^d, \\ S((\gamma_i^d)^*) &= S(\alpha_{i+d-1}^*)S(\alpha_{i+d-2}^*)\cdots S(\alpha_{i+1}^*)S(\alpha_i^*) \\ &= (-1)^{d+1} q^{d(d-1)(t+s)}(\gamma_{-i-d}^d)^*, \end{aligned}$$

then $S(I_d) \subseteq I_d$.

(2) We firstly show that $\Pi_0^1(\Gamma, I_d)$ has a basis

$$\{\gamma_i^u(\gamma_i^v)^* | u, v, i \in \mathbb{Z}_d\} \text{ and } \ \mathrm{dim}\Pi_0^1(\Gamma, \mathrm{I_d}) = \mathrm{d}^3.$$

- any nonzero monomial $g \in \Pi_0^1(\Gamma, I_d)$ is a linear combination of $\{\gamma_i^u(\gamma_i^v)^* | u, v, i \in \mathbb{Z}_d\}$,
- $\{\gamma_i^u(\gamma_i^v)^* | u, v, i \in \mathbb{Z}_d\}$ is linear independent: in fact,

$$0 = \sum_{u,v,i \in \mathbb{Z}_d} c_i^{u,v} \gamma_i^u (\gamma_i^v)^* = \sum_{l=0}^{2(d-1)} \sum_{\substack{u,v,i \in \mathbb{Z}_d, \\ u+v=l}} c_i^{u,v} \gamma_i^u (\gamma_i^v)^*,$$

$\{\gamma_i^u(\gamma_i^v)^* | u, v, i \in \mathbb{Z}_d, u + v = l\}$ have different sources and targets. Hence $c_i^{u,v} = 0$, and independent.

As for $\Pi_a^m(\Gamma, I_d)$, by Corollary 3.6 in [Crawley-Boevey W. Holland M.P. Noncommutative deformations of Kleinian singularities. Duke Mathematical Journal, 1998, 92(3): 605-635,] $\Pi_a^m(\Gamma)$ is a PBW-deformation of $\Pi_0^1(\Gamma)$. So $\Pi_a^m(\Gamma, I_d)$ is a PBW-deformation of $\Pi_0^1(\Gamma, I_d)$. Therefore, the statements in (2) hold for(deformed) preprojective algebra $\Pi_a^m(\Gamma)$.

Theorem

There is a Hopf isomorphism $\widetilde{\varphi} : \Pi_a^m(\Gamma, I_d) \to \overline{U}_q(sl_2^*, \kappa)$. *In particular, when* $a = 0$ *and* $m = 1$, *there is a Hopf isomorphism* $\widetilde{\varphi} : \Pi_0^1(\Gamma, I_d) \to \overline{U}_q(sl_2^*)$.

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2.4 Realization of $\overline{U}_q(sl_2^*, \kappa)$ by (deformed) preprojective algebra $\Pi_a^m(\Gamma)$

Theorem

There is a Hopf isomorphism $\widetilde{\varphi}: \Pi_a^m(\Gamma, I_d) \to \overline{U}_q(sl_2^*, \kappa)$. *In particular, when* $a = 0$ *and* $m = 1$, *there is a Hopf isomorphism* $\widetilde{\varphi}: \Pi_0^1(\Gamma, I_d) \to \overline{U}_q(sl_2^*)$.

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Proof. Step 1. Construct a surjective map $\varphi : \mathbb{C}\Gamma \longrightarrow \overline{U}_q(sl_2^*, \kappa)$.
Define a pair of maps $\varphi_0 : \Gamma_0 \longrightarrow \overline{U}_q(sl_2^*, \kappa)$ and $\varphi_1 : \Gamma_1 \longrightarrow \overline{U}_q(sl_2^*, \kappa)$ by setting

$$\varphi_0(s_l) = \epsilon_l, \quad \varphi_1(\alpha_l) = E\epsilon_{l+1}, \quad \varphi_1(\alpha_l^*) = F\epsilon_l$$

for each $l \in \mathbb{Z}_d$. It is easy to check that φ_0, φ_1 satisfy

$$\begin{cases} \sum\limits_{l \in \mathbb{Z}_d} \varphi_0(s_l) = 1, \\ \varphi_0(s_l)^2 = \varphi_0(s_l), \\ \varphi_0(s_k)\varphi_0(s_l) = 0 \text{ for } k \neq l, \\ \varphi_1(\alpha_l) = \varphi_0(s_l)\varphi_1(\alpha_l)\varphi_0(s_{l+1}), \\ \varphi_1(\alpha_l^*) = \varphi_0(s_{l+1})\varphi_1(\alpha_l^*)\varphi_0(s_l). \end{cases}$$

By the universal property of path algebra $\mathbb{C}\Gamma$, there exists a unique algebra homomorphism $\varphi : \mathbb{C}\Gamma \longrightarrow \overline{U}_q(sl_2^*, \kappa)$ such that

$$\varphi(s_l) = \varphi_0(s_l), \quad \varphi(\alpha_l) = \varphi_1(\alpha_l), \quad \text{and} \quad \varphi(\alpha_l^*) = \varphi_1(\alpha_l^*).$$

On the other hand, since

$$K = \sum_{l \in \mathbb{Z}_d} q^{-2l} \epsilon_l, \quad E = E \sum_{l \in \mathbb{Z}_d} \epsilon_l, \quad F = F \sum_{l \in \mathbb{Z}_d} \epsilon_l,$$

then $\varphi : \mathbb{C}\Gamma \longrightarrow \overline{U}_q(sl_2^*, \kappa)$ is surjective.

Step 2. Prove that $\widetilde{\varphi}:\Pi_a^m(\Gamma,I_d)\cong\overline{U}_q(sl_2^*,\kappa)$ as algebras.
Let $\mathcal{I}_d$ be the ideal

$$\left\langle \sum_{i\in\mathbb{Z}_d}\left(\alpha_i^*\alpha_i-\alpha_{i-1}\alpha_{i-1}^*-a(q^{-2mi}-q^{2mi})s_i\right),\gamma_i^d,(\gamma_i^d)^*\middle| i\in\mathbb{Z}_d\right\rangle$$

of $\mathbb{C}\Gamma$. Then $\Pi_a^m(\Gamma,I_d)=\mathbb{C}\Gamma/\mathcal{I}_d$. One can check that $\varphi(\mathcal{I}_d)=0$, i.e., $\mathcal{I}_d\subseteq\mathrm{Ker}\varphi$. On the other hand, we have proved that

$$\mathrm{dim}\overline{U}_q(sl_2^*,\kappa)=\mathrm{dim}\Pi_a^m(\Gamma,I_d)=d^3.$$

Hence $\mathrm{Ker}\varphi=\mathcal{I}_d$. By fundamental homomorphism theorem of algebras, $\varphi:\mathbb{C}\Gamma\longrightarrow\overline{U}_q(sl_2^*,\kappa)$ can induce a unique algebra isomorphism $\widetilde{\varphi}:\Pi_a^m(\Gamma,I_d)\to\overline{U}_q(sl_2^*,\kappa)$.

Step 2. Prove that $\widetilde{\varphi} : \Pi_a^m(\Gamma, I_d) \cong \overline{U}_q(sl_2^*, \kappa)$ as Hopf algebras.

In the following we only need to prove that $\widetilde{\varphi} : \Pi_a^m(\Gamma, I_d) \to \overline{U}_q(sl_2^*, \kappa)$ satisfies

$$\begin{cases} \Delta_{\overline{U}}\widetilde{\varphi}(x) = (\widetilde{\varphi} \otimes \widetilde{\varphi})\Delta_{\Pi}(x), \\ \varepsilon_{\Pi}(x) = \varepsilon_{\overline{U}}\widetilde{\varphi}(x), \\ \widetilde{\varphi}S_{\overline{U}}(x) = S_{\Pi}\widetilde{\varphi}(x) \end{cases}$$

for any $x \in \Gamma_0 \cup \Gamma_1 = \{s_l, \alpha_l, \alpha_l^* | l \in \mathbb{Z}_d\}$.

Remark

$$
\begin{array}{ccc}
\overline{U}_q(sl_2^*) & \xrightarrow{\text{PBW}-\text{deformation}} & \overline{U}_q(sl_2^*, \kappa) \\
\widetilde{\varphi} \Big\uparrow \cong & & \widetilde{\varphi} \Big\uparrow \cong \\
\Pi_0^1(\Gamma, I_d) & \xrightarrow{\text{PBW}-\text{deformation}} & \Pi_m^a(\Gamma, I_d)
\end{array}
$$

Proposition

Let M be a finite dimensional simple $\overline{U}_q(sl_2^*)$-module. Then $\dim(M) = 1$, and the module structure on $M = \mathbb{C}v_0$ can be given as follows:

$$Kv_0 = q^l v_0, \quad Ev_0 = Fv_0 = 0, \tag{11}$$

where $l \in \{0, 1, \cdots, d-1\}$ when n is odd and $l \in \{0, 2, \cdots, 2(d-1)\}$ when n is even..

Proof.

Since $\overline{U}_q(sl_2^*)$ is basic, then each simple $\overline{U}_q(sl_2^*)$-module is one-dimensional. Assume that $M = \mathbb{C}v_0$. It is clear that $Kv_0 = \lambda v_0$ and $Ev_0 = Fv_0 = 0$. Since $K^d = 1$, then $\lambda^d = 1$. Therefore, we conclude that $\lambda \in \{1, q, \cdots, q^{d-1}\}$ when $n = d$ is odd and $\lambda \in \{1, q^2, \cdots, q^{2(d-1)}\}$ when $n = 2d$ is even. □

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Definition

Let M be a finite dimensional representation of $\overline{U}_q(sl_2^*)$.
(1) The linear space

$$M_\lambda = \{v \in M | Kv = \lambda v\},$$

i.e., the eigenspace of K acting on M for the eigenvalue λ, is called a weight space of M.
(2) If M is the direct sum of its weight spaces, then we call M a weight representation of $\overline{U}_q(sl_2^*)$.
(3) Let M be a finite dimensional weight representation of $\overline{U}_q(sl_2^*)$, and denote by

$$\Lambda_M = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$$

the set of all the mutually different eigenvalues of K acting on M. We call Λ_M the weight set of M.

Definition

(1) Let $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be a subset of $\mathbb{C}^*$. If there exists a $\lambda \in \Lambda$ such that

$$\Lambda = \left\{\lambda, q^2\lambda, \cdots, q^{2(m-1)}\lambda\right\},$$

then we call Λ a q^2-chain.

(2) Let M be a finite dimensional weight representation of $\overline{U}_q(sl_2^*)$. If its weight set $\Lambda_M = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ is a q^2-chain, we call M a q^2-chain representation of $\overline{U}_q(sl_2^*)$.

Let M be a finite dimensional representation of $\overline{U}_q(sl_2^*)$. The action of the generator K on M can be considered as a linear transformation $K : M \longrightarrow M$. Since $K^d = 1$, then M can always be decomposed as the direct sum of the eigenspaces of K, i.e.,

$$M = \bigoplus_{\lambda \in \Lambda_M} M_\lambda,$$

where Λ_M consisting of all the eigenvalues of K is contained in the following set

$$\Lambda_{q^2} = \left\{1, q^2, q^4, \cdots, q^{2(d-2)}, q^{2(d-1)}\right\}$$

whenever n is odd or even.

Theorem

(1) *Each finite dimensional representation* M *of* $\overline{U}_q(sl_2^*)$ *is a weight representation.*

(2) *Each finite dimensional indecomposable representation* M *of* $\overline{U}_q(sl_2^*)$ *is a* q^2*-chain representation, where* Λ_M *is a* q^2*-chain with* $\Lambda_M \subseteq \Lambda_{q^2}$.

(3) *Each finite dimensional representation of* $\overline{U}_q(sl_2^*)$ *can be decomposed as the direct sum of some indecomposable* q^2*-chain representations of* $\overline{U}_q(sl_2^*)$.

Recall that $\Pi_0^1(\Gamma, I_d) = \mathbb{C}\Gamma/\mathcal{I}_d$, where

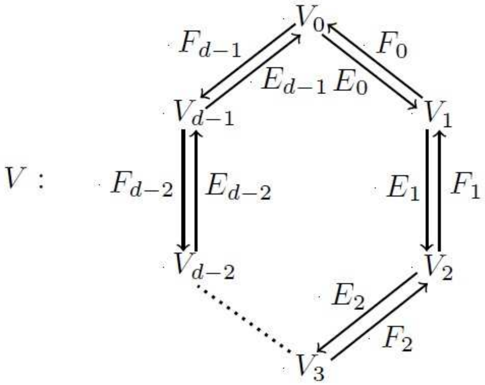
$$\mathcal{I}_d = \left\langle \sum_{i\in\mathbb{Z}_d} \left(\alpha_i^*\alpha_i - \alpha_{i-1}\alpha_{i-1}^*\right), \gamma_i^d, (\gamma_i^d)^* \middle| i \in \mathbb{Z}_d \right\rangle.$$

Denote by $\mathbf{rep}\Pi_0^1(\Gamma, I_d)$ the category of finite dimensional representations of $\Pi_0^1(\Gamma, I_d)$. Let $\mathcal{R}_\Gamma$ be the arrow ideal of $\mathbb{C}\Gamma$. Then the ideal $\mathcal{I}_d$ is an admissible ideal of $\mathbb{C}\Gamma$ because

$$0 = \mathcal{R}_\Gamma^d \subseteq \mathcal{I}_d \subseteq \mathcal{R}_\Gamma^2.$$

So we can always identify the finite dimensional representations of $\Pi_0^1(\Gamma, I_d)$ with those of the bound quiver $(\Gamma, \mathcal{I}_d)$.

In other words, each finite dimensional representation $V = (V_i, E_i, F_i)_{i \in \mathbb{Z}_d}$ of $\Pi_0^1(\Gamma, I_d)$ can be given as follows



where the matrix $E_i, F_i (i \in \mathbb{Z}_d)$ satisfy

$$\begin{cases} F_i E_i = E_{i-1} F_{i-1} \ (i \in \mathbb{Z}_d), \\ E_{i+d-1} \cdots E_{i+1} E_i = 0 \ (i \in \mathbb{Z}_d), \\ F_i F_{i+1} \cdots F_{i+d-1} = 0 \ (i \in \mathbb{Z}_d). \end{cases} \tag{12}$$

For any two objects $V = (V_i, E_i^V, F_i^V)_{i \in \mathbb{Z}_d}$ and $W = (W_i, E_i^W, F_i^W)_{i \in \mathbb{Z}_d}$ in $\mathbf{rep}\Pi_0^1(\Gamma, I_d)$, one has

$$f = (f_i)_{i \in \mathbb{Z}_d} \in \mathrm{Hom}_{\Pi_0^1(\Gamma, I_d)}(V, W)$$

such that

$$E_i^W f_i = f_{i+1} E_i^V, \ F_i^W f_{i+1} = f_i F_i^V \text{ for } i \in \mathbb{Z}_d.$$

Define a functor $\overline{\Omega}: \mathbf{rep}\Pi_0^1(\Gamma, I_d) \longrightarrow \mathbf{rep}\overline{U}_q(sl_2^*)$ as follows:

$$\begin{array}{rcl} \overline{\Omega}: \mathbf{rep}\Pi_0^1(\Gamma, I_d) & \longrightarrow & \mathbf{rep}\overline{U}_q(sl_2^*) \\ V = (V_i, E_i^V, F_i^V) & \longmapsto & \overline{\Omega}(V) \\ V \xrightarrow{f=(f_i)} W & \longmapsto & \overline{\Omega}(V) \xrightarrow{\overline{\Omega}(f)} \overline{\Omega}(W), \end{array}$$

where as a vector space $\overline{\Omega}(V) = \bigoplus_{i \in \mathbb{Z}_d} V_i$ and the action of $\overline{U}_q(sl_2^*)$ on $\Omega(V)$ is given by

$$\begin{cases} Kv = q^{2i}v, \\ Ev = E_i^V(v), \\ Fv = F_{i-1}^V(v) \end{cases}$$

for any $v \in V_i$, while $\overline{\Omega}(f) = \bigoplus_{i \in \mathbb{Z}_d} f_i$.

Define a functor $\overline{\Omega}^{-1} : \mathbf{rep}\overline{U}_q(sl_2^*) \longrightarrow \mathbf{rep}\Pi_0^1(\Gamma, I_d)$ as follows:

$$\begin{aligned}
\overline{\Omega}^{-1} : \mathbf{rep}\overline{U}_q(sl_2^*) &\longrightarrow \mathbf{rep}\Pi_0^1(\Gamma, I_d) \\
M &\longmapsto \overline{\Omega}^{-1}(M) \\
M \xrightarrow{f} N &\longmapsto \overline{\Omega}^{-1}(M) \xrightarrow{\overline{\Omega}^{-1}(f)} \overline{\Omega}^{-1}(N),
\end{aligned}$$

where $\overline{\Omega}^{-1}(M) := V = (V_i, E_i^V, F_i^V)_{i\in\mathbb{Z}_d}$ is given by

$$\begin{cases}
V_i = M_{q^{2i}}, \\
E_i^V = M_{q^{2i}} \xrightarrow{E} M_{q^{2(i+1)}}, \\
F_i^V = M_{q^{2(i+1)}} \xrightarrow{F} M_{q^{2i}}
\end{cases}$$

for $i \in \mathbb{Z}_d$, while $\overline{\Omega}^{-1}(f) := (g_i)_{i\in\mathbb{Z}_d}$ with g_i the restriction of f on $M_{q^{2i}}$.

Theorem

The functor $\overline{\Omega} : \mathbf{rep}\Pi_0^1(\Gamma, I_d) \longrightarrow \mathbf{rep}\overline{U}_q(sl_2^*)$ *is an equivalence of categories.*

The Hopf algebra isomorphism $\widetilde{\varphi} : \Pi_0^1(\Gamma, I_d) \longrightarrow \overline{U}_q(sl_2^*)$ we obtained can naturally induce an equivalence of categories

$$\begin{array}{rcl} \Omega_{\widetilde{\varphi}} : \mathbf{rep}\Pi_0^1(\Gamma, I_d) & \longrightarrow & \mathbf{rep}\overline{U}_q(sl_2^*) \\ M & \longmapsto & \Omega_{\widetilde{\varphi}}(M) = M \\ M \xrightarrow{f} N & \longmapsto & \Omega_{\widetilde{\varphi}}(f) = M \xrightarrow{f} N, \end{array}$$

where $\Omega_{\widetilde{\varphi}}(M) = M$ is a representation of $\overline{U}_q(sl_2^*)$ with the action of $\overline{U}_q(sl_2^*)$ on M given by

$$\begin{array}{rcl} \cdot : \overline{U}_q(sl_2^*) \otimes M & \longrightarrow & M \\ u \otimes m & \longmapsto & u \cdot m = \widetilde{\varphi}^{-1}(u)m. \end{array}$$

On the other hand, there exists a natural equivalence of categories

$$\begin{array}{rcl} \widetilde{\Omega}: \mathbf{rep}(\Gamma, \mathcal{I}_d) & \longrightarrow & \mathbf{rep}\Pi_0^1(\Gamma, I_d) \\ V = (V_i, E_i^V, F_i^V)_{i\in\mathbb{Z}_d} & \longmapsto & \widetilde{\Omega}(V) = \bigoplus_{i\in\mathbb{Z}_d} V_i \\ V \overset{f=(f_i)_{i\in\mathbb{Z}_d}}{\longrightarrow} W & \longmapsto & \widetilde{\Omega}(V) \overset{\widetilde{\Omega}(f)=\bigoplus_{i\in\mathbb{Z}_d} f_i}{\longrightarrow} \widetilde{\Omega}(W), \end{array}$$

where for any $v = (v_i)_{i\in\mathbb{Z}_d} \in \widetilde{\Omega}(V)$ and any path $w \in \Gamma$, the action of $\Pi_0^1(\Gamma, I_d)$ on $\widetilde{\Omega}(V)$ can be given by

$$(wv)_k = \begin{cases} \delta_{ik} v_i, & \text{if } w = s_i, \\ \delta_{jk}\psi_{\beta_l}\cdots\psi_{\beta_2}\psi_{\beta_1}(v_i), & \text{if } w = \beta_l\cdots\beta_2\beta_1 : i \to j. \end{cases}$$

with

$$\psi_{\beta_r} = \begin{cases} E_s^V, & \text{if } \beta_r = \alpha_s \text{ for some } s \in \mathbb{Z}_d, \\ F_s^V, & \text{if } \beta_r = \alpha_s^* \text{ for some } s \in \mathbb{Z}_d. \end{cases}$$

Corollary

As a functor, $\overline{\Omega}\widetilde{\Omega} = \Omega_{\widetilde{\varphi}}\widetilde{\Omega}$, i.e.,

$$\begin{array}{ccc}
\mathbf{rep}(\Gamma, \mathcal{I}_d) & \xrightarrow{\quad\widetilde{\Omega}\quad} & \mathbf{rep}\Pi_0^1(\Gamma, I_d) \\
\downarrow{\scriptstyle \widetilde{\Omega}} & & \downarrow{\scriptstyle \Omega_{\widetilde{\varphi}}} \\
\mathbf{rep}\Pi_0^1(\Gamma, I_d) & \xrightarrow{\quad\overline{\Omega}\quad} & \mathbf{rep}\overline{U}_q(sl_2^*)
\end{array}$$

Definition

(1) Let $V = (V_i, E_i, F_i)_{i\in\mathbb{Z}_d}$ be a finite dimensional representation in the category $\mathbf{rep}(\Gamma, \mathcal{I}_d)$. If V is indecomposable and $V_i \neq 0$ for all $i \in \mathbb{Z}_d$, then we call V a primitive representation in $\mathbf{rep}(\Gamma, \mathcal{I}_d)$.

(2) Let M be an indecomposable representation in the category $\mathbf{rep}\overline{U}_q(sl_2^*)$. If the weight set Λ_M of M can be given as follows

$$\Lambda_M = \{1, q^2, \cdots, q^{2i}, \cdots, q^{2l}\}$$

for some integer $l \in \mathbb{Z}_d$, then we call M a primitive representation of $\overline{U}_q(sl_2^*)$.

Definition

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$$\Lambda_M = \{1, q^2, \cdots, q^{2i}, \cdots, q^{2l}\}$$

for some integer $l \in \mathbb{Z}_d$, then we call M a primitive representation of $\overline{U}_q(sl_2^*)$.

Assume that M is a primitive representation of $\overline{U}_q(sl_2^*)$ with weight set

$$\Lambda_M = \{1, q^2, \cdots, q^{2i}, \cdots, q^{2l}\}$$

for some $l \in \mathbb{Z}_d$. Then $M = \bigoplus_{i=0}^{l} M_{q^{2i}}$. For each $0 \leq i \leq l$, assume that $v_{i1}, v_{i2}, \cdots, v_{in_i}$ is a basis of $M_{q^{2i}}$, we obtain a basis

$$B_M = (v_{01}, v_{02}, \cdots, v_{0n_0}, \cdots, v_{i1}, v_{i2}, \cdots, v_{in_i}, \cdots, v_{l1}, v_{l2}, \cdots, v_{ln_l})$$

of M. Considered as linear transformations on M, the generators K, E, F of $\overline{U}_q(sl_2^*)$ acting on the basis B_M are respectively corresponding to the following three matrix $\mathcal{K}, \mathcal{E}, \mathcal{F}$, i.e.,

$$
\begin{aligned}
\mathcal{K} &= \begin{pmatrix} I_{n_0} & & & \\ & q^2 I_{n_1} & & \\ & & \ddots & \\ & & & q^{2l} I_{n_l} \end{pmatrix}, \\
\mathcal{E} &= \begin{pmatrix} 0 & & & \delta_{l,d-1}\mathcal{E}_l \\ \mathcal{E}_0 & 0 & & \\ & & \ddots & \\ & & \mathcal{E}_{l-1} & 0 \end{pmatrix}, \\
\mathcal{F} &= \begin{pmatrix} 0 & \mathcal{F}_0 & & \\ & 0 & & \\ & & \ddots & \mathcal{F}_{l-1} \\ \delta_{l,d-1}\mathcal{F}_l & & & 0 \end{pmatrix},
\end{aligned}
$$

where I_{n_i} is the $n_i \times n_i$ identity matrix, $\mathcal{E}_i$ is a $n_{i+1} \times n_i$ matrix, $\mathcal{F}_i$ is a $n_i \times n_{i+1}$ matrix, and $\mathcal{E}_i, \mathcal{F}_i$ $(0 \leq i \leq l)$ satisfy

$$\begin{cases} \delta_{l,d-1}\mathcal{E}_l\mathcal{F}_l = \mathcal{F}_0\mathcal{E}_0, \\ \mathcal{E}_i\mathcal{F}_i = \mathcal{F}_{i+1}\mathcal{E}_{i+1} \ (0 \leq i \leq l-2), \\ \mathcal{E}_{l-1}\mathcal{F}_{l-1} = \delta_{l,d-1}\mathcal{F}_l\mathcal{E}_l, \\ \mathcal{E}_i\mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_0\mathcal{E}_{d-1}\cdots\mathcal{E}_{i+1} = 0 \ (0 \leq i \leq d-1), \\ \mathcal{F}_i\mathcal{F}_{i+1}\cdots\mathcal{F}_{d-1}\mathcal{F}_0\mathcal{F}_1\cdots\mathcal{F}_{i-1} = 0 \ (0 \leq i \leq d-1). \end{cases} \tag{13}$$

Theorem

Assume that M is a d-dimensional primitive representation of $\overline{U}_q(sl_2^*)$ with $\Lambda_M = \{q^{2i} | i \in \mathbb{Z}_d\}$. Then M is isomorphic to a d-dimensional primitive representation $\overline{L}_{\mathcal{E},\mathcal{F}}$ defined by

$$\begin{cases} KB_M = B_M\mathcal{K}, \\ EB_M = B_M\mathcal{E}, \\ FB_M = B_M\mathcal{F}, \end{cases}$$

where

$$\begin{cases} \mathcal{E}_i\mathcal{F}_i = 0 \ (i \in \mathbb{Z}_d), \\ \mathcal{E}_i + \mathcal{F}_i = 1 \ (i \in \mathbb{Z}_d), \end{cases} \quad \text{or} \begin{cases} \mathcal{E}_i\mathcal{F}_i = 0 \ (i \in \mathbb{Z}_d), \\ \exists | i_0 \in \mathbb{Z}_d \text{ s.t. } \mathcal{E}_{i_0} = \mathcal{F}_{i_0} = 0, \\ \mathcal{E}_i + \mathcal{F}_i = 1 \ (i \in \mathbb{Z}_d \setminus \{i_0\}). \end{cases}$$

Thank you!