Tilting modules, dominant dimensions and Brauer-Schur-Weyl duality

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目录

Invariant theory

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In each case, the first fundamental theorem (FFT) provides generators and the second fundamental theorem (SFT) describes all relations among the generators.

Schur-Weyl duality between $\mathrm{GL}(V)$ and $\mathfrak{S}_n$

Let $m, n \in \mathbb{N}$ and V be a m-dimensional vector space over a field F of characteristic 0. Let $\mathfrak{S}_n$ be the symmetric group.

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Let $m, n \in \mathbb{N}$ and V be a m-dimensional vector space over a field F of characteristic 0. Let $\mathfrak{S}_n$ be the symmetric group.

There is a left action of $\mathfrak{S}_n$ on $V^{\otimes n}$ which commutes with the left action of $\mathrm{GL}(V)$. Let φ, ψ be the natural algebra homomorphisms:

$$\varphi: F\mathfrak{S}_n \to \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n}),$$
$$\psi: F\mathrm{GL}(V) \to \mathrm{End}_{F\mathfrak{S}_n}(V^{\otimes n}).$$

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Theorem

(SFT) *If $m < n$, we have*

$$\mathrm{Ann}_{F\mathfrak{S}_n}\left(V^{\otimes n}\right) = \langle y_{m+1} \rangle,$$

where $y_{m+1} = \sum_{w \in \mathfrak{S}_{m+1}} (-1)^{\ell(w)} w$ is the anti-symmetrizer of Young subgroup $\mathfrak{S}_{(m+1, 1^{n-m-1})}$.

History of Schur-Weyl duality for type A

The integral version: see [Carter-Lusztig, Math. Zeit. **136** (1974), 193-242], [Green, LNM **830** (1980)], etc.

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The integral quantum version of FFT and SFT: still unknown.

Topological categorification

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Reshetikhin-Turaev proved the following in [Comm. Math. Phys. **127** (1990), 1-26]:

Theorem

Let $\mathcal{R}(K)$ be the category of directed ribbon graphs over the field K. Let $\mathcal{C}$ be a K-linear ribbon category with a twist and left duality. Given any object V in $\mathcal{C}$, denote $T(V)$ the tensor subcategory of $\mathcal{C}$ generated by V and its dual object. Then there exists a unique braided tensor functor $F : \mathcal{R}(K) \to T(V)$, which preserves left duality and twist.

Classical case

Using the category of Brauer diagrams, a symmetric braided strict monoidal category, Lehrer and Zhang in [J. Euro. Math. Soc. **17** (2015), 2311-2351] provided a unified description for the different formulations of the fundamental theorems for classical cases.

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Furthermore, they gave recursive formulaes of the SFTs.

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In [Xiao-Yang-Zhang, Sci. China Math. **63** (2020), 689-700], we introduced the diagram category of framed tangles and provided a recursive formula of the SFT of quantized symplectic groups.

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Remark (1)

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Remark (1)

The diagram category of framed tangles is a pivotal braided strict monoidal category.

Remark (2)

If we assume the SFT of quantized orthogonal groups is available (as shown by Lehrer and Zhang), then we can obtain the the SFT of quantized special orthogonal groups.

Tilting modules

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Let V be the natural representation of $\mathrm{GL}(V)$. Then there is a natural epimorphism

$$\pi_\lambda : V^{\otimes n} \twoheadrightarrow \nabla(\lambda)$$

for each co-Weyl module (also called Schur module) with $\lambda \vdash n$ and $\ell(\lambda) \leq \dim(V)$.

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$$\Delta(\lambda) \hookrightarrow V^{\otimes n} \twoheadrightarrow \nabla(\lambda)$$

forms a perfect pair.

Tilting modules

We first give a formal generalization of the Carter-Lusztig's results.

Theorem

Let A be a finite dimensional standardly stratified algebra and T be a tilting A-module. Suppose that there is an integer r such that for any $\lambda \in \Lambda^+$, there is an embedding $\iota_\lambda : \Delta(\lambda) \hookrightarrow T^{\oplus r}$ as well as an epimorphism $\pi_\lambda : T^{\oplus r} \twoheadrightarrow \overline{\nabla}(\lambda)$ as A-modules, then T is a faithful module over A and A has the double centraliser property with respect to T. That is,

$$A = \mathrm{End}_{\mathrm{End}_A(T)}(T).$$

Tilting modules

The above Theorem is related to an old problem:

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Let A be a quasi-hereditary algebra and T be a tilting A-module. When A has the double centraliser property with respect to T?

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When the T-dominant dimension of A (regular A-module) is at least two, the problem is known (see [Tachikawa, LNM **351** (1973)] and [Auslander-Solberg, Comm. Algebra **21** (1993), 3081-3097]).

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Maybe one complete answer for the above problem is faithfulness.

Tilting modules

Theorem (Hu-Xiao, Trans. AMS. to appear)

Let A be a quasi-hereditary algebra with a simple preserving duality $\circ$. Let T be a faithful tilting A-module. Then

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In particular, the T-dominant dimension of A is at least two.

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In particular, the T-dominant dimension of A is at least two.

Using this Theorem, we can recover many known double centralizer properties or simplify the proof of the corresponding Schur-Weyl dualities in even integral situation.

Application to Schur-Weyl duality

It is possible to simplify the proof of Schur-Weyl dualities. General speaking, we have two algebras A, B and an (A, B)-bimodule M. By a Schur-Weyl duality between A and B on the bimodule M we mean that the following two canonical maps:

$$\varphi : A \to \mathrm{End}_B(M), \quad \psi : B \to \mathrm{End}_A(M),$$

are both surjective.

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If we can show that the image of φ in $\mathrm{End}_B(M)$ is a quasi-hereditary algebra with a simple preserving duality, then then the surjectivity of φ will follow from the surjectivity of ψ, i.e.

$$A \twoheadrightarrow \mathrm{Im}(\varphi) = \mathrm{End}_{\mathrm{End}_{\mathrm{Im}(\varphi)}(M)}(M)$$

$$= \mathrm{End}_{\mathrm{End}_A(M)}(M) = \mathrm{End}_B(M).$$

A byproduct

We affirmatively answer a question of Mazorchuk and Stroppel ([J. Reine. Angew. Math. **616** (2008), 131-165]) on the existence of minimal basic tilting module T for which A has the double centralizer property.

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Theorem (Hu-Xiao, Trans. AMS. to appear)

Let A be a quasi-hereditary algebra with a simple preserving duality. Then there exists a unique faithful basic tilting module T such that

1. $A = \mathrm{End}_{\mathrm{End}_A(T)}(T)$; *and*
2. *if T' is another faithful tilting module satisfying $A = \mathrm{End}_{\mathrm{End}_A(T')}(T')$, then T must be a direct summand of T'.*

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Our second attempt still fails !!!

谢谢！