Some properties of non-pointed Hopf algebras generalized from pointed ones

Kangqiao Li

Department of Mathematics, Nanjing University

Southwest University, Chongqing. August, 2021

Outline

Abstract

There were a number of classic results on pointed Hopf algebras. Some of them might be generalized to **non-pointed cases**, with the methods of so-called multiplicative and primitive matrices. The aim of this talk is to introduce these methods and results.

Specifically, for a non-pointed Hopf algebra with the (dual Chevalley property):

1) The coradical filtration is initially determined by matrices mentioned above;
2) There is an annihilation polynomial for the antipode;
3) We show a formula on the products between the link-indecomposable components.

Related works

Some results are joint works with Prof. Shenglin Zhu and Prof. Gongxiang Liu.

The results introduced in this talk are selected from the following articles:

[1] Kangqiao Li, Shenglin Zhu, *On the exponent of finite-dimensional non-cosemisimple Hopf algebras*, Comm. Algebra 47 (2019), no. 11, 4476-4495.

[2] Kangqiao Li, Gongxiang Liu, *On the antipode of Hopf algebras with the dual Chevalley property*, J. Pure Appl. Algebra 226 (2022), no. 3, 106871.

[3] Kangqiao Li, *Note on invariance and finiteness for the exponent of Hopf algebras*, Comm. Algebra, published online.

[4] Kangqiao Li, *The link-indecomposable components of Hopf algebras and their products*, preprint (in revision).

All these articles could be found on **arXiv**.

Coalgebras and Hopf algebras

- In this talk, all vector spaces, coalgebras, and Hopf algebras are assumed to be over a field $\Bbbk$.
- **Coalgebra.** A coalgebra H is a triple (H, Δ, ε), where H is a $\Bbbk$-vector space, and $\Delta : C \to C \otimes C$, $\varepsilon : C \to \Bbbk$ are linear maps, such that following diagrams both commute:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H\otimes H \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ H\otimes H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H\otimes H\otimes H \end{array} \qquad \begin{array}{ccccc} & \overset{\cong}{\swarrow} & H & \overset{\cong}{\searrow} & \\ & & \downarrow{\scriptstyle\Delta} & & \\ \Bbbk\otimes H & \xleftarrow[\varepsilon\otimes\mathrm{id}]{} & H\otimes H & \xrightarrow[\mathrm{id}\otimes\varepsilon]{} & H\otimes\Bbbk \end{array}$$

  Δ and ε are called the comultiplication and the counit, respectively.
- **Hopf algebra.** Suppose that (H, m, u) is an $\Bbbk$-algebra, and (H, Δ, ε) is a $\Bbbk$-coalgebra. H is said to be a Hopf algebra over $\Bbbk$, if
  (1) Δ and ε are both algebra maps (H is called a **bialgebra**);
  (2) There is a linear map $S : H \to H$, such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

  hold on H. S is called the antipode.

Pointed coalgebras and Hopf algebras

- **Grouplike and primitive element.** Let H be a coalgebra.
  (1) $g \in H$ is said to be grouplike, if
  $$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1.$$
  The set of all the grouplike elements of H is denoted by $G(H)$.
  (2) Suppose $g, h \in H$ are grouplike. $x \in H$ is said to be (g,h)-primitive, if
  $$\Delta(x) = g \otimes x + x \otimes h, \quad (\varepsilon(x) = 0).$$
  The set of all the (g,h)-primitive elements of H is denoted by $P_{g,h}(H)$.
- **Fact 1.** Each 1-dimensional (simple) coalgebra is spanned by a unique grouplike element.
- **Fact 2.** Suppose H is a Hopf algebra. Then $G(H)$ is a group, which would be finite if H is moreover finite-dimensional.
- **Pointed coalgebra** & **Hopf algebra.** A coalgebra (or Hopf algebra) H is said to be pointed, if its coradical is exactly $\Bbbk G(H)$.

Outline

Multiplicative matrices over a coalgebra H

- **Multiplicative matrix.** (Manin 1988)
  A matrix $\mathcal{G} = (g_{ij})_{n\times n}$ over H is said to be multiplicative, if for each $1 \leq i,j \leq n$,
  $$\Delta(g_{ij}) = \sum_{k=1}^{n} g_{ik} \otimes g_{kj}, \qquad \varepsilon(g_{ij}) = \delta_{ij}.$$
- **Basic fact.** Suppose C is a simple coalgebra over an algebraically closed field $\Bbbk$. Then C has a linear basis $\{c_{ij} \mid 1 \leq i,j \leq r\}$ such that $(c_{ij})_{r\times r}$ is a multiplicative matrix (which is called a basic multiplicative matrix of C).

Multiplicative matrices over a coalgebra H

- **Multiplicative matrix.** (Manin 1988)
  A matrix $\mathcal{G} = (g_{ij})_{n\times n}$ over H is said to be multiplicative, if for each $1 \leq i,j \leq n$,
  $$\Delta(g_{ij}) = \sum_{k=1}^{n} g_{ik} \otimes g_{kj}, \quad \varepsilon(g_{ij}) = \delta_{ij}.$$
- **Basic fact.** Suppose C is a simple coalgebra over an algebraically closed field $\Bbbk$. Then C has a linear basis $\{c_{ij} \mid 1 \leq i,j \leq r\}$ such that $(c_{ij})_{r\times r}$ is a multiplicative matrix (which is called a basic multiplicative matrix of C).
- **Primitive matrix.** Let $\mathcal{C}_{r\times r}$ and $\mathcal{D}_{s\times s}$ be basic multiplicative matrices over H. A matrix $\mathcal{X} = (x_{ij})_{r\times s}$ over H is said to be $(\mathcal{C},\mathcal{D})$-primitive, if for each $1 \leq i \leq r$ and $1 \leq j \leq s$,
  $$\Delta(x_{ij}) = \sum_{k=1}^{r} c_{ik} \otimes x_{kj} + \sum_{l=1}^{s} x_{il} \otimes d_{lj}, \quad (\varepsilon(x_{ij}) = 0).$$
- **Remark.** $\mathcal{X}$ is $(\mathcal{C},\mathcal{D})$-primitive, if and only if $\begin{pmatrix} \mathcal{C} & \mathcal{X} \\ 0 & \mathcal{D} \end{pmatrix}$ is multiplicative.

Grouplike elements v.s. multiplicative matrices

- The definition of a multiplicative matrix $\mathcal{G} = (g_{ij})_{n\times n}$ might be written as

$$\Delta(\mathcal{G}) = \mathcal{G} \mathbin{\widetilde{\otimes}} \mathcal{G} \quad \text{and} \quad \varepsilon(\mathcal{G}) = I_n,$$

where $\mathcal{G} \mathbin{\widetilde{\otimes}} \mathcal{G} = (\sum_{k=1}^{n} g_{ik} \otimes g_{kj})_{n\times n}$ is a matrix over $H \otimes H$.

- **Observation.** $\widetilde{\otimes}$ is a "associative binary operation" on matrices over vector spaces.

Grouplike elements v.s. multiplicative matrices

- The definition of a multiplicative matrix $\mathcal{G} = (g_{ij})_{n\times n}$ might be written as

$$\Delta(\mathcal{G}) = \mathcal{G} \,\widetilde{\otimes}\, \mathcal{G} \quad \text{and} \quad \varepsilon(\mathcal{G}) = I_n,$$

where $\mathcal{G} \,\widetilde{\otimes}\, \mathcal{G} = (\sum_{k=1}^{n} g_{ik} \otimes g_{kj})_{n\times n}$ is a matrix over $H \otimes H$.

- **Observation.** $\widetilde{\otimes}$ is a "associative binary operation" on matrices over vector spaces.
- **Basic case: Span simple coalgebras.** When $\overline{\mathbb{k}} = \mathbb{k}$, any simple coalgebra C has a basic multiplicative matrix $\mathcal{C}$. The uniqueness could be described as follows:

Fact ([4, Lemma 2.4])

Let C be a simple coalgebra with a basic multiplicative matrix $\mathcal{C}$. Then the followings are equivalent:
(1) $\mathcal{D}$ is also a basic multiplicative matrix of C;
(2) $\mathcal{D} \sim \mathcal{C}$, which means that $\mathcal{D} = L\mathcal{C}L^{-1}$ for some matrix L over $\mathbb{k}$.

A "non-basic case" of this fact is described in [4, Proposition 2.6].

Grouplike elements v.s. multiplicative matrices

Suppose H is a bialgebra. Recall that $G(H)$ is a monoid with the unit element 1.

- **The monoid of multiplicative matrices.** The set of all multiplicative matrices (over H) is closed under the Kronecker product $\odot$.

Fact ([4, Lemma 2.7])

Suppose $\mathcal{A} = (a_{ij})_{r\times r}$ and $B = (b_{ij})_{s\times s}$ be multiplicative matrices over a bialgebra H. Then the following $rs \times rs$ matrix is multiplicative:

$$\mathcal{A} \odot \mathcal{B} := \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathcal{B} & \cdots & a_{nn}\mathcal{B} \end{pmatrix}.$$

Grouplike elements v.s. multiplicative matrices

Suppose H is a bialgebra. Recall that $G(H)$ is a monoid with the unit element 1.

- **The monoid of multiplicative matrices.** The set of all multiplicative matrices (over H) is closed under the Kronecker product $\odot$.

Fact ([4, Lemma 2.7])

Suppose $\mathcal{A} = (a_{ij})_{r\times r}$ and $B = (b_{ij})_{s\times s}$ be multiplicative matrices over a bialgebra H. Then the following $rs \times rs$ matrix is multiplicative:

$$\mathcal{A} \odot \mathcal{B} := \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathcal{B} & \cdots & a_{nn}\mathcal{B} \end{pmatrix}.$$

Suppose H is moreover a Hopf algebra with antipode S.

- **Fact (Inverse).** For any multiplicative matrix $\mathcal{G}$ over H, the matrix $S(\mathcal{G})^{\mathrm{T}}$ is also multiplicative, and $S(\mathcal{G})\mathcal{G} = \mathcal{G}S(\mathcal{G}) = I$ holds over H.

Grouplike elements v.s. multiplicative matrices

Suppose H is a bialgebra. Recall that $G(H)$ is a monoid with the unit element 1.

- **The monoid of multiplicative matrices.** The set of all multiplicative matrices (over H) is closed under the Kronecker product $\odot$.

Fact ([4, Lemma 2.7])

Suppose $\mathcal{A} = (a_{ij})_{r\times r}$ and $B = (b_{ij})_{s\times s}$ be multiplicative matrices over a bialgebra H. Then the following $rs \times rs$ matrix is multiplicative:

$$\mathcal{A} \odot \mathcal{B} := \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathcal{B} & \cdots & a_{nn}\mathcal{B} \end{pmatrix}.$$

Suppose H is moreover a Hopf algebra with antipode S.

- **Fact (Inverse).** For any multiplicative matrix $\mathcal{G}$ over H, the matrix $S(\mathcal{G})^{\mathrm{T}}$ is also multiplicative, and $S(\mathcal{G})\mathcal{G} = \mathcal{G}S(\mathcal{G}) = I$ holds over H.

What if multiplicative matrices are considered to be basic?

Dual Chevalley property

A Hopf algebra H is said to have the dual Chevalley property, if its coradial H_0 is a Hopf subalgebra.

- **Obvious fact.** Pointed Hopf algebras have the dual Chevalley property.

Dual Chevalley property

A Hopf algebra H is said to have the dual Chevalley property, if its coradial H_0 is a Hopf subalgebra.

- **Obvious fact.** Pointed Hopf algebras have the dual Chevalley property.

- Let H be a Hopf algebra with the dual Chevalley property.

A corollary of [4, Lemma 2.7(2) and Proposition 2.6(2)]

Suppose $\mathcal{C}$ and $\mathcal{D}$ are basic multiplicative matrices over H. Then

$$\mathcal{C} \odot \mathcal{D} \sim \mathrm{diag}(\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t),$$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t$ are basic multiplicative matrices.

Dual Chevalley property

A Hopf algebra H is said to have the dual Chevalley property, if its coradial H_0 is a Hopf subalgebra.

- **Obvious fact.** Pointed Hopf algebras have the dual Chevalley property.

- Let H be a Hopf algebra with the dual Chevalley property.

A corollary of [4, Lemma 2.7(2) and Proposition 2.6(2)]

Suppose $\mathcal{C}$ and $\mathcal{D}$ are basic multiplicative matrices over H. Then

$$\mathcal{C} \odot \mathcal{D} \sim \operatorname{diag}(\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t),$$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t$ are basic multiplicative matrices.

This is essentially the same thing as the product of characters for semisimple Hopf algebras.

Conclusion: How to generalize pointed Hopf algebras

- We conclude the facts as follows:

Pointed Hopf algebras	**Non-pointed Hopf algebras**	**Sufficient condition**
Simple subcoalgebras are spanned by grouplike elements.	Simple subcoalgebras are spanned by entries of basic multiplicative matrices.	The base field $\Bbbk$ is algebraically closed.
Grouplike elements are closed under the multiplication.	Kronecker products of basic multiplicative matrices are similar to block diagonal matrices with entries as basic multiplicative ones.	The dual Chevalley property.
Grouplike elements has inverses.	Basic multiplicative matrices has inverses as the transpose of basic ones.	The antipode S is bijective.
Grouplike elements are pairwise (via inverses).	Basic multiplicative matrices are pairwise.	Involutory, i.e. $S^2 =$ id.

- Note that the dual Chevalley property implies that S is bijective.

Conclusion: How to generalize pointed Hopf algebras

- We conclude the facts as follows:

Pointed Hopf algebras	**Non-pointed Hopf algebras**	**Sufficient condition**
Simple subcoalgebra $\Bbbk g$ is spanned by $g \in G(H)$	Simple subcoalgebra C is spanned by basic $\mathcal{C}$	$\overline{\Bbbk} = \Bbbk$.
$\forall g, h \in G(H)$, $gh \in G(H)$.	$\forall$ basic $\mathcal{C}, \mathcal{D}$, $\mathcal{C} \odot \mathcal{D} \sim \mathrm{diag}(\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$.	${H_0}^2 \subseteq H_0$.
$\forall g \in G(H)$, $g^{-1} \in G(H)$.	$\forall$ basic $\mathcal{C}$, $S(\mathcal{C})^{\mathrm{T}}$ is basic.	S is bijective.
$g \leftrightarrow g^{-1}$ in $G(H)$, and $(g^{-1})^{-1} = g$.	$\mathcal{C} \leftrightarrow S(\mathcal{C})^{\mathrm{T}}$, and $S(S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} = \mathcal{C}$.	$S^2 = \mathrm{id}$.

Outline

Two notions of exponent and finiteness

- There are two notions of the exponent for Hopf algebras.

Definitions (Kashina 1999; Etingof and Gelaki, 1999)

Let H be a Hopf algebra with bijective antipode S.

$$\begin{aligned}\exp_0(H) &:= \{n \geq 1 \mid \forall h \in H, \sum h_{(1)} h_{(2)} \cdots h_{(n)} = \varepsilon(h)1\};\\ \exp(H) &:= \{n \geq 1 \mid \forall h \in H, \sum h_{(1)} S^{-2}(h_{(2)}) \cdots S^{-2n+2}(h_{(n)}) = \varepsilon(h)1\}.\end{aligned}$$

- **Known results.** (Etingof and Gelaki, 1999)
  (1) When H is semisimple and cosemisimple, $\exp(H) = \exp_0(H) \mid \dim(H)^3$;
  (2) When H is finite-dimensional in positive characteristic, $\exp(H) < \infty$.

- **Question.** (Etingof and Gelaki, 2002)
  Is $\exp(H)$ infinite when H is non-semisimple in characteristic 0?

Two notions of exponent and finiteness

- **Our answers:**

Proposition ([3, Proposition 4.1])

Let H be a finite-dimensional Hopf algebra in positive characteristic. Then $\exp_0(H) < \infty$.

- Suppose that H is a non-cosemisimple Hopf algebra with the dual Chevalley property.

Theorem ([1, Theorem 4.1], [3, Proposition 4.2 and Theorem 4.11])

(1) If char $\Bbbk = 0$, then $\exp_0(H) = \infty$, and meanwhile $\exp(H) = \infty$;
(2) If H is finite-dimensional in characteristic $p > 0$, then

$$\exp_0(H) \mid Np^M \quad \text{and} \quad \exp(H) \mid N'p^M,$$

where $N := \exp_0(H_0)$, $N := \operatorname{lcm}(\exp_0(H_0), \exp(H_0))$, and p^M is not less than the Loewy length of H (i.e. $H_{p^M-1} = H$).

Primitive elements v.s. primitive matrices (1)

- Denote the set of all the simple subcoalgebras of a coalgebra H by $\mathcal{S}$. Let $\{e_C\}_{C\in\mathcal{S}} \subseteq H^*$ be a family of coradical orthonormal idempotents, satisfying

$$e_C\mid_D = \delta_{C,D}\varepsilon_D, \quad e_Ce_D = \delta_{C,D}e_C \quad (\forall C, D \in \mathcal{S}), \quad \text{and} \quad \sum_{C\in\mathcal{S}} e_C = \varepsilon.$$

- **Remark.** The existence of such $\{e_C\}_{C\in\mathcal{S}}$ is affirmed by Radford in 1978.
- We use notations: ${}^{C}h^{D} := e_D \rightharpoonup h \leftharpoonup e_C = \sum\langle e_C, h_{(1)}\rangle h_{(2)}\langle e_D, h_{(3)}\rangle$, etc.

Primitive elements v.s. primitive matrices (1)

- Denote the set of all the simple subcoalgebras of a coalgebra H by $\mathcal{S}$. Let $\{e_C\}_{C\in\mathcal{S}} \subseteq H^*$ be a family of coradical orthonormal idempotents, satisfying

$$e_C \mid_D = \delta_{C,D}\varepsilon_D, \quad e_C e_D = \delta_{C,D} e_C \quad (\forall C, D \in \mathcal{S}), \quad \text{and} \quad \sum_{C\in\mathcal{S}} e_C = \varepsilon.$$

- **Remark.** The existence of such $\{e_C\}_{C\in\mathcal{S}}$ is affirmed by Radford in 1978.
- We use notations: ${}^C h^D := e_D \rightharpoonup h \leftharpoonup e_C = \sum \langle e_C, h_{(1)}\rangle h_{(2)} \langle e_D, h_{(3)}\rangle$, etc.

Proposition (Taft and Wilson, 1974)

Let H be a pointed coalgebra. Then for any $g, h \in G(H)$:
(1) $P_{g,h}(H) = \Bbbk(g-h) \oplus {}^g{H_1}^h$, if $g \neq h$;
(2) ${}^g{H_1}^g = \Bbbk g \oplus P_{g,g}(H)$.

- **Note:** The "difference" between ${}^g{H_1}^h$ and $P_{g,h}(H)$ are contained in H_0.

- **Remark.** When H is pointed, we may write $H_1 = \bigoplus_{g,h\in G(H)} {}^g{H_1}^h$. Thus H_1 is spanned by grouplike and primitive elements.

Primitive elements v.s. primitive matrices (1)

- The definition of a $(\mathcal{C},\mathcal{D})$-primitive matrix $\mathcal{X}=(X_{ij})_{r\times s}$ might be written as
$$\Delta(\mathcal{X})=\mathcal{C}\,\widetilde{\otimes}\,\mathcal{X}+\mathcal{X}\,\widetilde{\otimes}\,\mathcal{D}.$$
- Let H be a coalgebra. A generalized Taft-Wilson proposition could be:

Proposition ([1, Theorem 3.1])

Suppose that $C,D\in\mathcal{S}$ with basic multiplicative matrices $\mathcal{C}_{r\times r},\mathcal{D}_{s\times s}$, respectively.

(1) If $C\neq D$, then for any $x\in {}^{C}H_1{}^{D}$, there exist rs $(\mathcal{C},\mathcal{D})$-primitive matrices
$$\mathcal{X}^{(i',j')}=\left(x_{ij}^{(i',j')}\right)_{r\times s}\qquad (1\leq i'\leq r,\ 1\leq j'\leq s),$$
such that $x=\sum_{i=1}^{r}\sum_{j=1}^{s}x_{ij}^{(i,j)}$;

(2) If $C=D$ and assume $\mathcal{C}=\mathcal{D}$, then for any $x\in {}^{C}H_1{}^{C}$, there exist r^2 $(\mathcal{C},\mathcal{C})$-primitive matrices
$$\mathcal{X}^{(i',j')}=\left(x_{ij}^{(i',j')}\right)_{r\times r}\qquad (1\leq i',j'\leq r),$$
such that $x-\sum_{i,j=1}^{r}x_{ij}^{(i,j)}\in C$.

Primitive elements v.s. primitive matrices (1)

Proposition ([1, Theorem 3.1])

Suppose that $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}_{r\times r}, \mathcal{D}_{s\times s}$, respectively.

(1) If $C \neq D$, then for any $x \in {}^{C}H_1{}^{D}$, there exist rs $(\mathcal{C},\mathcal{D})$-primitive matrices

$$\mathcal{X}^{(i',j')} = \left(x_{ij}^{(i',j')}\right)_{r\times s} \qquad (1 \leq i' \leq r,\ 1 \leq j' \leq s),$$

such that $x = \sum_{i=1}^{r}\sum_{j=1}^{s} x_{ij}^{(i,j)}$;

(2) If $C = D$ and assume $\mathcal{C} = \mathcal{D}$, then for any $x \in {}^{C}H_1{}^{C}$, there exist r^2 $(\mathcal{C},\mathcal{C})$-primitive matrices

$$\mathcal{X}^{(i',j')} = \left(x_{ij}^{(i',j')}\right)_{r\times r} \qquad (1 \leq i',j' \leq r),$$

such that $x - \sum_{i,j=1}^{r} x_{ij}^{(i,j)} \in C$.

- **Conclusion.** ${}^{C}H_1{}^{D}$ is spanned by entries of $(\mathcal{C},\mathcal{D})$-primitive matrices as well as some elements in $C + D$. If $\overline{\Bbbk} = \Bbbk$ and consider $H_1 = \bigoplus_{C,D\in\mathcal{S}} {}^{C}H_1{}^{D}$, then H_1 is spanned by entries of basic multiplicative and primitive matrices.

Primitive elements v.s. primitive matrices (2)

Let H be a Hopf algebra with antipode S.

- **Fact.** For $x \in \underline{P_{g,1}(H)}$, direct computations follow that

$$S^2(x) = g^{-1}xg \quad \text{and} \quad S^{2n}(x) = g^{-N}xg^N = x,$$

where $N := \exp(G(H)) \mid \exp(H_0)$.

- **Generalization:**

Proposition ([2, Lemmas 3.5 and 3.6])

Suppose that $\mathcal{C}$ is a basic multiplicative matrix, and $\mathcal{X}$ is a $(\mathcal{C}, 1)$-primitive matrix.
(1) $S^2(\mathcal{X}) = ((S(\mathcal{C})\mathcal{X})^{\mathrm{T}} S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}$;
(2) If H has the dual Chevalley property, then

$$S^{2N}(\mathcal{X}) = \mathcal{X},$$

where $N := \exp(H_0)$ is the exponent of the Hopf subalgebra H_0.

- **Remark.** ([3, Proposition 4.10]) If H is finite-dimensional, then there exists a $(\mathcal{C}, 1)$-primitive matrix $\mathcal{X}'$ such that $S^2(\mathcal{X}') = q\mathcal{X}'$, for some Nth root $q \in \Bbbk$ of unity.

An annihilation polynomial for the antipode

Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote the Loewy length of H by $L := \min\{l \geq 0 \mid H_{l-1} = H\}$.

- **Known result.** (Taft and Wilson, 1974) Suppose H is pointed. Denote $N := \exp(G(H))$. Then $(S^{2N} - \mathrm{id})^{L-1}=0$ holds on H.
- **Generalization:**

Theorem ([2, Theorem 3.1])

Denote $N := \exp(H_0)$. Then $(S^{2N} - \mathrm{id})^{L-1}=0$ holds on H.

An annihilation polynomial for the antipode

Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote the Loewy length of H by $L := \min\{l \geq 0 \mid H_{l-1} = H\}$.

- **Known result.** (Taft and Wilson, 1974) Suppose H is pointed. Denote $N := \exp(G(H))$. Then $(S^{2N} - \mathrm{id})^{L-1}=0$ holds on H.
- **Generalization:**

Theorem ([2, Theorem 3.1])

Denote $N := \exp(H_0)$. Then $(S^{2N} - \mathrm{id})^{L-1}=0$ holds on H.

- **Consequences:**

Theorem ([2, Corollary 3.3 and Theorem 4.3])

Suppose that char $\Bbbk = 0$. Then
(1) The composition order of S^2 divides $\exp(H_0)$;
(2) Particularly if $\Bbbk = \mathbb{C}$, then the quasi-exponent (introduced by Etingof and Gelaki in 2002) of H is exactly $\exp(H_0)$.

Link relation and link-indecomposable component

Let H be a coalgebra. Denote the set of all its simple subcoalgebras by $\mathcal{S}$.

- **Link relation on $\mathcal{S}$.** (Montgomery, 1995; Radford, 2012)
  Suppose $C, D \in \mathcal{S}$.
  (1) C and D are said to be directly linked, if $C + D \subsetneq C \wedge D + D \wedge C$;
  (2) C and D are said to be linked, if there exist $n \geq 0$ and $C = E_0, E_1, \cdots, E_n = D \in \mathcal{S}$, such that E_i and E_{i+1} are directly linked for $0 \leq i < n$.
- **Remark.** The link relation is an equivalence relation on $\mathcal{S}$.

- **Link-indecomposable component.** A link-indecomposable component of H is a maximal subcoalgebra H', such that any two simple subcoalgebras of H' are linked.

Lemma (Montgomery, 1995)

Any coalgebra H is presented uniquely as a direct sum $H = \bigoplus_i H_{(i)}$ of indecomposable subcoalgebras, where each $H_{(i)}$ is exactly a link-indecomposable component of H.

Link-indecomposable components of pointed Hopf algebras

Let H be a pointed Hopf algebra.

- Let $H_{(g)}$ denote the link-indecomposable component containing $g \in G(H)$. Then:

Theorem (Montgomery, 1995)

(1) $H_{(1)}$ is a Hopf subalgebra;
(2) For any $g, h \in G(H)$, $H_{(g)}H_{(h)} \subseteq H_{(gh)}$ and $S(H_{(g)}) \subseteq H_{(g^{-1})}$ hold;
(3) H is (left and right) free over $H_{(1)}$. Specifically, $H_{(g)} = gH_{(1)} = H_{(1)}g$ for each $g \in G(H)$, and

$$H = \bigoplus_{g \in G(H)/G(H_{(1)})} gH_{(1)}.$$

Note that $G(H_{(1)})$ is a normal subgroup of $G(H)$.

- **Consequences.** H is (left and right) faithfully flat over the normal Hopf subalgebra $H_{(1)}$.
- **Aim.** Generalize (1), (2) and the faithful flatness to the case with the dual Chevalley property.

Link-indecomposable components of Hopf algebras with the dual Chevalley property

Let H be a Hopf algebra with the dual Chevalley property over an algebraically closed field $\mathbb{k}$. Note that S is then bijective.

- Let $H_{(C)}$ denote the link-indecomposable component containing $C \in \mathcal{S}$. Then:

Theorem ([4, Theorem 3.16 and Corollary 3.17])

(1) $H_{(1)}$ is a Hopf subalgebra;

(2) For any $C, D \in \mathcal{S}$,

$$H_{(C)}H_{(D)} \subseteq \sum_{E\in\mathcal{S},\ E\subseteq CD} H_{(E)}$$

and $S(H_{(C)}) \subseteq H_{(S(C))}$ hold;

(3) H is (left and right) faithfully flat over the Hopf subalgebra $H_{(1)}$.

- **Remark.** ([4, Proposition 3.13])

  A weaker sufficient condition for (1) is $(H_{(1)})_0{}^3 \subseteq H_0$, instead of the dual Chevalley property. (Example)

Further results (in revision)

Recall that when H is pointed:

Theorem (Montgomery, 1995)

(3) H is (left and right) free over $H_{(1)}$. Specifically, $H_{(g)} = gH_{(1)} = H_{(1)}g$ for each $g \in G(H)$, and

$$H = \bigoplus_{g \in G(H)/G(H_{(1)})} gH_{(1)}.$$

Let H be a Hopf algebra with the dual Chevalley property over an arbitrary field $\mathbb{k}$.

Theorem

(4) H is not always free over $H_{(1)}$. However, $H_{(C)} = CH_{(1)} = H_{(1)}C$ holds for each $C \in \mathcal{S}$, and

$$H = \bigoplus_{C} CH_{(1)},$$

where C runs over arbitrary chosen representatives with respect to “some equivalence relation”.

A sufficient condition for the link relation

Let H be a coalgebra.

- A sufficient condition for two simple subcoalgebras to be linked is Item (1) of the following:

Lemma ([4, Proposition 3.8(2) and Lemma 4.1])

Suppose that

$$\mathcal{G} := \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix}$$

is a (block) multiplicative matrix over H, where $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ are basic multiplicative matrices for $C_1, C_2, \cdots, C_t$ respectively.
(1) If any entry of $\mathcal{X}_{1t}$ does not belong to H_0, then C_1 and C_t are linked;
(2) If $C_1, C_2, \cdots, C_t$ are linked, then all the entries of $\mathcal{G}$ belong to this link-indecomposable component.

- **Remark.** Item (2) could be used to find the (link-)indecomposable decomposition of H. See the example below.

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

Now we work on an algebraically closed field $\Bbbk$ of characteristic 0.

- **Infinite-dimensional Taft algebra.** (Lu, Wu and Zhang, 2007)
  As an algebra, $T_\infty(2,1,-1)$ is generated by g and x with relations:
  $$g^2 = 1, \quad xg = -gx.$$
  Then $T_\infty(2,1,-1)$ becomes a Hopf algebra with comultiplication, counit and antipode given by
  $$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$
  $$S(g) = g, \quad S(x) = gx.$$
- **Remark.** Infinite-dimensional Taft algebras $T_\infty(n, v, \xi)$ are among the class of affine prime regular Hopf algebras of GK-dimension one.
- **Aim:** Let us consider its finite dual $T_\infty(2,1,-1)^\circ$.

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

Example (Brown, Couto and Jahn, 2021; Li and Liu, 2021)

As an algebra, $T_\infty(2,1,-1)^\circ$ is generated by ψ_λ $(\lambda \in \Bbbk)$, ω, E_2, E_1 with relations

$$\psi_{\lambda_1}\psi_{\lambda_2} = \psi_{\lambda_1+\lambda_2}, \quad \psi_0 = 1, \quad \omega^2 = 1, \quad E_1^2 = 0,$$
$$\omega\psi_\lambda = \psi_\lambda\omega, \quad E_2\omega = \omega E_2, \quad E_1\omega = -\omega E_1,$$
$$E_2\psi_\lambda = \psi_\lambda E_2, \quad E_1\psi_\lambda = \psi_\lambda E_1, \quad E_1E_2 = E_2E_1$$

for all $\lambda, \lambda_1, \lambda_2 \in \Bbbk$. The coalgebra structure and antipode are given by:

$$\Delta(\omega) = \omega\otimes\omega, \quad \Delta(E_1) = 1\otimes E_1 + E_1\otimes\omega,$$
$$\Delta(E_2) = 1\otimes E_2 + E_1\otimes\omega E_1 + E_2\otimes 1,$$
$$\Delta(\psi_\lambda) = (\psi_\lambda\otimes\psi_\lambda)(1\otimes 1 + \lambda E_1\otimes\omega E_1),$$
$$\varepsilon(\omega) = \varepsilon(\psi_\lambda) = 1, \quad \varepsilon(E_1) = \varepsilon(E_2) = 0,$$
$$S(\omega) = \omega, \quad S(E_1) = \omega E_1, \quad S(E_2) = -E_2, \quad S(\psi_\lambda) = \psi_{-\lambda},$$

for $\lambda \in \Bbbk$.

Note that $\{\psi_\lambda\omega^jE_2^sE_1^l \mid \lambda\in\Bbbk,\ 0\le j,l\le 1,\ s\in\mathbb{N}\}$ is a linear basis.

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

- Note that $\{\psi_\lambda \omega^j E_2^s E_1^l \mid \lambda \in \Bbbk,\ 0 \leq j, l \leq 1,\ s \in \mathbb{N}\}$ is a linear basis.
- **Fact.** ([4, Proposition 4.9]) Following matrices over $\underline{H := T(2,1,-1)^\circ}$ are multiplicative:

  (1) 1 and ω;

  (2) $\mathcal{E} := \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & \omega & \omega E_1 \\ 0 & 0 & 1 \end{pmatrix} \implies \Bbbk 1$ and $\Bbbk\omega$ are linked;

  (3) $\mathcal{E}^{\odot s}$ for all $s \geq 1 \implies$ Their entries belong to $H_{(1)}$;

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

- Note that $\{\psi_\lambda \omega^j E_2^s E_1^l \mid \lambda \in \Bbbk,\ 0 \leq j, l \leq 1,\ s \in \mathbb{N}\}$ is a linear basis.
- **Fact.** ([4, Proposition 4.9]) Following matrices over $\underline{H := T(2,1,-1)^\circ}$ are multiplicative:

  (1) 1 and ω;

  (2) $\mathcal{E} := \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & \omega & \omega E_1 \\ 0 & 0 & 1 \end{pmatrix} \implies \Bbbk 1$ and $\Bbbk\omega$ are linked;

  (3) $\mathcal{E}^{\odot s}$ for all $s \geq 1 \implies$ Their entries belong to $H_{(1)}$;

  (4) For each $\lambda \in \Bbbk^*$, $\mathcal{C}_\lambda := \begin{pmatrix} \psi_\lambda & \lambda\psi_\lambda E_1 \\ \psi_\lambda \omega E_1 & \psi_\lambda \omega \end{pmatrix}$ is basic for the simple subcoalgebra C_λ;

  (5) For each $\lambda \in \Bbbk^*$, $\mathcal{E}^{\odot s} \odot \mathcal{C}_\lambda$ for all $s \geq 1 \implies$ Entries belong to $H_{(C_\lambda)}$.

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

- Note that $\{\psi_\lambda \omega^j E_2^s E_1^l \mid \lambda \in \Bbbk,\ 0 \leq j,l \leq 1,\ s \in \mathbb{N}\}$ is a linear basis.
- **Fact.** ([4, Proposition 4.9]) Following matrices over $\underline{H := T(2,1,-1)^\circ}$ are multiplicative:
  (1) 1 and ω;
  (2) $\mathcal{E} := \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & \omega & \omega E_1 \\ 0 & 0 & 1 \end{pmatrix} \implies \Bbbk 1$ and $\Bbbk\omega$ are linked;
  (3) $\mathcal{E}^{\odot s}$ for all $s \geq 1 \implies$ Their entries belong to $H_{(1)}$;
  (4) For each $\lambda \in \Bbbk^*$, $\mathcal{C}_\lambda := \begin{pmatrix} \psi_\lambda & \lambda\psi_\lambda E_1 \\ \psi_\lambda \omega E_1 & \psi_\lambda \omega \end{pmatrix}$ is basic for the simple subcoalgebra C_λ;
  (5) For each $\lambda \in \Bbbk^*$, $\mathcal{E}^{\odot s} \odot \mathcal{C}_\lambda$ for all $s \geq 1 \implies$ Entries belong to $H_{(C_\lambda)}$.
- **Conclusion.** Since every basis element appears as some entry above, the link-indecomposable decomposition for $H := T(2,1,-1)^\circ$ is then

$$H = H_{(1)} \oplus \left(\bigoplus_{\lambda \in \Bbbk^*} H_{(C_\lambda)} \right).$$

Example: Decomposition of $T_\infty(2,1,-1)^\circ$

Still denote $H := T_\infty(2,1,-1)^\circ$. Some observations:

- $T_\infty(2,1,-1)^\circ$ does not have the dual Chevalley property, because

$$C_\lambda C_{-\lambda} = \Bbbk\{1,\ \omega,\ E_1,\ \omega E_1\}.$$

  However, the weaker condition $(H_{(1)})_0{}^3 \subseteq H_0$ holds, and thus $H_{(1)}$ is still a Hopf subalgebra.

- More than $H_{(C_\lambda)} = C_\lambda H_{(1)} = H_{(1)} C_\lambda$, we could find that

$$H_{(C_\lambda)} = \psi_\lambda H_{(1)} \quad \text{and} \quad H_{(C_\lambda)} = H_{(1)}\psi_\lambda$$

  hold (also as left and right $H_{(1)}$-modules respectively).
  Thus H is free over the normal Hopf subalgebra $H_{(1)}$.

- Denote the Hopf algebra $H_{(1)}$ by $T_\infty(2,1,-1)^\bullet$. In fact the evaluation

$$\langle -,-\rangle : T_\infty(2,1,-1)^\bullet \otimes T_\infty(2,1,-1) \to \Bbbk$$

  is a non-degenerate Hopf pairing of GK-dimension one.

References

- K. Brown, M. Couto and A. Jahn, *The finite dual of commutative-by-finite Hopf algebras*. arXiv:2105.13874.
- P. Etingof and S. Gelaki, *On the exponent of finite-dimensional Hopf algebras*. Math. Res. Lett. 6 (1999), no. 2, 131-140.
- P. Etingof and S. Gelaki, *On the quasi-exponent of finite-dimensional Hopf algebras*. Math. Res. Lett. 9 (2002), no. 2-3, 277-287.
- Y. Kashina, *On the order of the antipode of Hopf algebras in* ${}^H_H\mathcal{YD}$. Comm. Algebra 27 (1999), no. 3, 1261-1273.
- K. Li and G. Liu, *The finite duals of affine prime regular Hopf algebras of GK-dimension one*. arXiv:2103.00495.
- D.-M. Lu, Q.-S. Wu and J.J. Zhang, *Homological integral of Hopf algebras*. Trans. Amer. Math. Soc. 359 (2007), no. 10, 4945-4975.
- Y.I. Manin, *Quantum Groups and Noncommutative Geometry*. Universitĺę de Montrĺęal, Centre de Recherches Mathĺęmatiques, Montreal, QC, 1988. vi+91 pp.

References

- S. Montgomery, *Indecomposable coalgebras, simple comodules, and pointed Hopf algebras*. Proc. Amer. Math. Soc. 123 (1995), no. 8, 2343-2351.
- D.E. Radford, *On the structure of commutative pointed Hopf algebras*. J. Algebra 50 (1978), no. 2, 284-296.
- D.E. Radford, *Hopf Algebras*. Series on Knots and Everything, 49. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. xxii+559 pp.
- E.J. Taft and R.L. Wilson, *On antipodes in pointed Hopf algebras*. J. Algebra 29 (1974), 27-32.

Thank you !