Skew Calabi-Yau property of Hopf Galois extensions

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Outline

Definitions

Let $\mathbb{k}$ be a field.

Definition 1.1

*A $\mathbb{k}$-algebra A is called **skew Calabi-Yau** (skew CY for short) of dimension d for some integer $d \geq 0$, if*

(i) *A is **homologically smooth**, that is, as an $A^e(:= A \otimes A^{op})$-module, A has a finitely generated projective resolution of finite length;*

(ii) $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A_{\mu_A}, & i = d \end{cases}$ *as A^e-modules, for some automorphism μ_A. This μ_A is called the **Nakayama automorphism** of A.*

If μ_A is inner, then A is called **Calabi-Yau**.*[Ginzburg, 2006]*

Examples

(1) Let X be a smooth Calabi-Yau variety(i.e., the canonical sheaf is trivial). Then $\mathbb{C}[X]$ is a CY-algebra;

(2) All Known Noetherian Hopf algebras with finite global dimension;

(3) Ore extention of a skew CY-algebra [Liu-Wang-Wu, 2012];

(4) Smash product of a skew CY-algebra by a skew CY Hopf algebra [Le Meur, 2019];

(5) Connected graded skew CY-algebras "$=$" Artin-Schelter regular algebras [Reys-Rogalski-Zhang, 2015].

Artin-Schelter Gorenstein Hopf algebra

Definition 1.2

A Hopf algebra $(H;\Delta,\varepsilon;S)$ is called **AS-Gorenstein**, *if*

(1) *H has finite injective dimension d;*

(2) $\mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \cong \mathrm{Ext}^i_{H^{op}}(\Bbbk_\varepsilon, H) \cong \begin{cases} 0, & i \neq d \\ \Bbbk, & i = d \end{cases}.$

If $\mathrm{gldim}(H) = d$, *then H is called* **AS-regular**.

Lu-Wu-Zhang, 2007

$\int^l_H := \mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H)$ is called the **left homological integral** of H.

Theorem 1.3 (Brown-Zhang, 2008)

Let H be a Noetherian Hopf algebra. Then H is d-dim skew CY if and only if H is d-dim AS-regular.

Noetherian Hopf algebras are always AS-Gorenstein?

Theorem 1.4 (Larson-Sweedler, 1969)

All finite dimensional Hopf algebras are 0-dim AS-Gorenstein.

Brown-Goodearl Conjecture

All Noetherian Hopf algebras has finite injective dimension.

Theorem 1.5 (Wu-Zhang, 2003)

Any Noetherian affine PI Hopf algebra is AS-Gorenstein.

Brown-Goodearl-Wu-Zhang Conjecture

All Noetherian Hopf algebras are AS-Gorenstein.

A criterion for homological smooth algberas to be skew Calabi-Yau

Let A be a homological smooth algebra.

Lemma 1.6

If $\Omega := \boldsymbol{R}\mathrm{Hom}_{A^e}(A, A^e)$ *is a perfect complex in* $\mathcal{D}(A)$ *and* $\mathcal{D}(A^{op})$*, then* Ω *is a two side tilting complex over* A*, that is,*

$$\Omega \overset{\boldsymbol{L}}{\otimes}_A \boldsymbol{R}\mathrm{Hom}_A(\Omega, A) \cong \boldsymbol{R}\mathrm{Hom}_A(\Omega, A) \overset{\boldsymbol{L}}{\otimes}_A \Omega \cong A \text{ in } \mathcal{D}(A^e).$$

Theorem 1.7

If $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ *for* $i \neq d$ *and* $\omega := \mathrm{Ext}^d_{A^e}(A, A^e)$ *is a finitely generated projective left and right* A*-module, then* A *has a* d*-dim* **Van den Bergh duality**, *i.e.,* ω *is an invertible* A*-*A*-bimodule. If moreover,* ${}_A\omega \cong {}_AA$ *or* $\omega_A \cong A_A$*, then* A *is skew CY.*

Quantum homogeneous space

Let B be a quantum homogeneous space of a Hopf algebra H, that is, B is right coideal subalgebra of H, H_B and ${}_BH$ are faithfully flat.

Lemma 1.8

If B is homological smooth, then for all i, as left H-modules,

$$H \otimes_B \operatorname{Ext}^i_B(B, B^e) \cong \operatorname{Ext}^i_B({}_\varepsilon \Bbbk, B) \otimes H.$$

Corollary 1.9 (Krähmer's question, 2012)

All homological smooth AS-regular quantum homogeneous spaces have Van den Bergh dualities.

Hopf Galois extension

Let H be a Hopf algebra, and B be a right H-comodule algebra.

Definition 2.1

*Then $A = B^{coH} \subset B$ is called an H-**Galois** extension if the Galois map $\beta : B \otimes_A B \to B \otimes H,\ b' \otimes b \mapsto \sum_{(b)} b' b_0 \otimes b_1$, is bijective.*

Theorem 2.2 (Schneider, 1990)

Then the following are equivalent.

- (i) $(- \otimes_A B, (-)^{coH})$ *is an equivalence between* $\mathcal{M}_A$ *and* $\mathcal{M}_B^H$.
- (ii) $(B \otimes_A -, (-)^{coH})$ *is an equivalence between* $\mathcal{M}_A$ *and* ${}_B\mathcal{M}^H$.
- (iii) ${}_AB$ *is faithfully flat,* $A \subseteq B$ *is an H-Galois extension.*
- (iv) B_A *is faithfully flat,* $A \subseteq B$ *is an H-Galois extension.*

H-module structure on Hochschild cohomology

Throughout, let H be a n-dim skew Calabi-Yau Hopf algebra, and $A \subseteq B$ be a faithfully flat H-Galois extension.

For any ${}_BM_B$, $\mathrm{HH}^0(A, M) = M^A := \{m \in M \mid am = ma, \ \forall\, a \in A\}$ is a right H-module via

$$m \leftharpoonup h := \sum \kappa^1(h) m \kappa^2(h)$$

where $\sum \kappa^1(h) \otimes_A \kappa^2(h) = \beta^{-1}(1_B \otimes h) \in B \otimes_A B$.

The H-module structure is induced on $\mathrm{Ext}^i_{A^e}(A, M) \cong \mathrm{HH}^i(A, M)$.

Stefan's spectral sequence

Theorem 2.3 (Stefan, 1995)

For any B^e-module M, we have that

$$\boldsymbol{R}\mathrm{Hom}_{B^e}(B, M) \cong \boldsymbol{R}\mathrm{Hom}_{H^{op}}(\Bbbk, \boldsymbol{R}\mathrm{Hom}_{A^e}(A, M)).$$

Corollary 2.4 (Stefan, 1995)

If A is separable and H is simisimple, then B is also separable.

Main Theorem

Proposition 2.5 (Liu-Wu-Zhu, 2012)

If A and H are homological smooth, then so is B.

Lemma 2.6

(1) $\mathrm{Ext}^d_{A^e}(A, B^e) \cong \omega \otimes_A B \otimes H$ *as right* $B \otimes H$*-modules.*

(2) $\mathrm{Ext}^d_{A^e}(A, B^e) \cong B \otimes_A \omega \otimes H$ *as left* $B \otimes H^{op}$*-modules.*

Theorem 2.7

(1) *If A has a d-dim Van den Bergh duality, then B also has a $(d+n)$-dim Van den Bergh duality.*

(2) *If A is d-dim skew CY, then B is $(d+n)$-dim skew CY.*

When H is finite dimensional

Proposition 3.1

Let H be a finite dimensional semisimple Hopf algebra, and $A \subseteq B$ be an H-Galois extension. If A is skew Calabi-Yau, then there exists a Nakayama automorphism μ_B of B such that

$$\mu_B|_A \text{ is a Nakayama automorphism of } A.$$

When A is finite dimensional

Theorem 3.2 (Yu, 2016)

Let H be a d-dim skew Calabi-Yau algebra, and $A = \Bbbk \subseteq B$ be an H-Galois extension. Then B is d-dim skew Calabi-Yau.

Suppose that A is a strongly separable algebra, i.e., there exists a symmetric separability idempotent $\sum x_i \otimes y_i \in A^e$ such that

$$\sum ax_i \otimes y_i = \sum x_i \otimes y_i a, \ \ \sum x_i y_i = 1, \ \ \sum x_i \otimes y_i = \sum y_i \otimes x_i.$$

Lemma 3.3 (Aguiar, 2000)

$T : (B \otimes B)^A \to B \otimes_A B$, $\sum x_i \otimes y_i \mapsto \sum y_i \otimes_A x_i$ *is an* " $\cong$ ".

Take $\sum \kappa^2(h) \otimes \kappa^1(h) = T^{-1} \circ \beta^{-1}(1_B \otimes h) \in (B \otimes B)^A$.

Theorem 3.4

If H is d-dim skew Calabi-Yau with $\int_H^l \cong \Bbbk_\chi$, then B is d-dim skew Calabi-Yau with a Nakayama automorphism μ, defined by

$$\mu : B \longrightarrow B, \quad b \mapsto \sum_{(b)} \chi(b_2)\kappa^2(S^{-2}b_1)\kappa^1(S^{-2}b_1)b_0.$$

Morita-Takeuchi equivalence and Galois extension

Definition 3.5

Two Hopf algebras are called Morita-Takeuchi equivalent, if their comodule categories are monoidally equivalent.

Theorem 3.6 (Schauenburg, 1996)

Let K, H are two Hopf algberas. Then K and H are Morita-Takeuchi equivalent if and only if there is a K-H-bigalois extension of $\mathbb{k}$.

Theorem 3.7 (Wang-Yu-Zhang, 2017)

Let K and H be two Morita-Takeuchi equivalent Hopf algebras. If H is skew CY of dimension d and K is homologically smooth, then K is skew CY of dimension d.

Question 3.8 (Bichon, 2013)

If K and H are two Morita-Takeuchi equivalent Hopf algebras, then $\mathrm{gldim}(H) = \mathrm{gldim}(K)$?

Question 3.9 (Wang-Yu-Zhang, 2017)

If H is homologically smooth, then K is homologically smooth?

An example

Let $\lambda \in \mathbb{k}$ be an n-th primitive root of unity,
$H = \mathbb{k}\langle g^{\pm}, x\rangle / \langle xg - \lambda gx, x^n + g^n - 1\rangle$ be the Liu Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x,$$

$K = \mathbb{k}\langle g^{\pm}, y\rangle / \langle yg - \lambda gy, y^n\rangle$ be the Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \Delta(y) = 1 \otimes y + y \otimes g,$$

$B = \mathbb{k}\langle g^{\pm}, t\rangle / \langle tg - \lambda gt, t^n + g^n\rangle$ be a K-H-bicomodule algebra via

$$\rho_H(g) = g \otimes g, \rho_H(t) = t \otimes 1 + g \otimes x, \rho_K(g) = g \otimes g, \rho_K(t) = g \otimes t + y \otimes 1.$$

Then B is an K-H-bigalois extension of $\mathbb{k}$, $\operatorname{gldim} H = 1$ and $\operatorname{gldim} K = +\infty$.

Proposition 3.10

Let K and H be two Morita-Takeuchi equivalent Noetherian Hopf algebras.

(1) *If* $\mathrm{r.gldim}(H) = d < +\infty$, *then* $\mathrm{injdim}({}_K K) = d$.

(2) *If H is d-dim AS-regular, then K is d-dim AS-Gorenstein.*

Question 3.11

(1) *Does* $\mathrm{injdim}({}_K K) = \mathrm{injdim}(H_H)$*?*

(2) *If H is d-dim AS-Gorenstein, then K is d-dim AS-Gorenstein?*

Proposition 3.12

Let K and H be two Morita-Takeuchi equivalent Noetherian affine PI Hopf algebras. Then $\mathrm{injdim}(K) = \mathrm{injdim}(H)$.

Cleft extension

Let H be a Hopf algebra, and B be a right H-comodule algebra. $A(:= B^{coH}) \subseteq B$ is called H-**cleft** if there exists a convolution invertible H-comodule map $\gamma : H \to B$. Assume that $\gamma(1) = 1$.

Proposition 3.13 (Doi-Takeuchi, 1986)

Then there is a crossed product action of H on A, given by

$$h \cdot a = \sum \gamma(h_1) a \gamma^{-1}(h_2), \ \ \forall\, a \in A, h \in H,$$

and a convolution invertible map $\sigma : H \otimes H \to A$ given by

$$\sigma(h, k) = \sum \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2 k_2), \ \ \forall\, h, k \in H.$$

Then B is isomorphic to the crossed product $A \#_\sigma H$.

H-actions

Definition 3.14

(1) *For any vector space V, we say that there is a left H-action "$\cdot$" on V, if there exists a $\Bbbk$-linear map $H \otimes V \longrightarrow V$, $h \otimes v \mapsto h \cdot v$ such that $1 \cdot v = v$ for any $v \in V$.*

(2) *Let U, V be two vector spaces with H-actions and $\alpha : U \to V$ be a $\Bbbk$-linear map. We say that α preserves the H-action, if $\alpha(h \cdot u) = h \cdot \alpha(u)$.*

Lemma 3.15

For any $n \in \mathbb{N}$, B^e-module X, there exists left H-actions "$\cdot$" on

(1) $\mathrm{Hom}_{A^e}(A^{\otimes n+1}, X)$ *via* $(h \cdot f)(a_0, \cdots, a_{n+1}) =$

$$\sum \gamma^{-1}(Sh_{n+4})f(Sh_{n+3} \cdot a_0 \otimes \cdots \otimes Sh_2 \cdot a_{n+1})\gamma(Sh_1);$$

(2) $\mathrm{Hom}_{A^e}(A^{\otimes n+1}, A^e)$ *via* $(h \cdot g)(a_0, \cdots, a_{n+1}) =$

$$\sum (\gamma^{-1}(Sh_{n+5}) \otimes \gamma(Sh_2))g(Sh_{n+4} \cdot a_0, \cdots, Sh_3 \cdot a_{n+1})$$

$$(\gamma^{-1}(S^2 h_{n+6}) \otimes \gamma(h_1)),$$

(3) $\mathrm{Hom}_{A^e}(A^{\otimes n+1}, A^e) \otimes_{A^e} B^e$ *via* $h \cdot (g \otimes_{A^e} (b \otimes b')) =$

$$\sum (h_2 \cdot g) \otimes_{A^e} (\gamma(S^2 h_3) b \otimes b' \gamma^{-1}(h_1)).$$

The differentials of $\mathrm{Hom}_{A^e}(\mathrm{B}_\bullet(A), A^e)$, $\mathrm{Hom}_{A^e}(\mathrm{B}_\bullet(A), B^e)$ and $\mathrm{Hom}_{A^e}(\mathrm{B}_\bullet(A), A^e) \otimes_{A^e} B^e$ preserve the H-action.

Lemma 3.16

The H-actions are induced on $\mathrm{Ext}^i_{A^e}(A, A^e)$, $\mathrm{Ext}^i_{A^e}(A, B^e)$, $\mathrm{Ext}^i_{A^e}(A, A^e) \otimes_{A^e} B^e$, such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) \otimes_{A^e} B^e \xrightarrow{\cong} \mathrm{Ext}^i_{A^e}(A, B^e)$$

preserves the H-action, and for all $x \in \mathrm{Ext}^i_{A^e}(A, B^e)$ and $h \in H$,

$$h \cdot x = x \leftharpoonup Sh.$$

Homological determinant

Suppose that A is a d-dim skew CY algebra with a Nakayama automorphism μ_A. Then there exists $e \in \omega := \operatorname{Ext}^d_{A^e}(A, A^e)$ such that $Ae = \omega$, $ea = \mu_A(a)e$ for any $a \in A$. Then there exists a morphism $\varphi \in \operatorname{Hom}(H, A)$, such that

$$h \cdot e = \varphi(h)e.$$

Lemma 3.17

φ has a left convolution inverse Hdet *defined by*

$$h \mapsto \sum_{(h)} \sigma^{-1}(h_5, S^{-1}h_4)h_6 \cdot (\varphi(S^{-1}h_3)\mu_A(\sigma(Sh_2, S^2h_1))).$$

Then $\operatorname{Hdet} : H \to A$ *is called the* **homological determinant** *of* $A = B^{coH}$ *with respect to* e.

Skew Calabi-Yau property for smash product

Theorem 3.18 (Le Meur, 2015)

If A and H are skew CY, then $A\#H$ is also skew CY, and

$$\mu_{A\#H}(a\#h) = \sum_{(h)} \mu_A(a)\,\mathrm{Hdet}(S^{-2}h_1)\#S^{-2}h_2\chi(h_3),$$

where $\int_H^l \cong \Bbbk_\chi$ is the left homological integral of H.

Theorem 3.19 (Yu-Zhang, 2016)

If H is skew CY, A is N-Koszul graded skew CY, then the graded crossed prduct $A\#_\sigma H$ is also skew CY.

Nakayama automorphism of cleft extensions

Theorem 3.20

Let H be a n-dim skew CY Hopf algebra, and $A \subseteq B$ be a cleft H-extension. If A is d-dim skew CY with a Nakayama automorphism, then B is also a skew CY algebra of dimension $n+d$ with Nakayama automorphism μ_B which is defined by

$$\mu_B(b) = \sum_{(h)} \mu_A(a) \, \mathrm{Hdet}(S^{-2}h_1) \# S^{-2}h_2 \chi(h_3),$$

where $\chi : H \to \Bbbk$ is given by $\int_H^l \cong \Bbbk_\chi$.

Thank You!