Representations of small quasi-quantum group $\widetilde{u_q}(\mathfrak{sl}_2)$

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2021Hopf 代数会议 (西南大学)，2021 年 8 月 24 日

Joint work with Huixiang Chen and Yinhuo Zhang

Contents

Motivation

1. We have studied Representations (Green rings) of Drinfeld doubles of Taft algebra.
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Quasion: What is relation between the representation categories of small quantum group and small quasi-quantum group?

Green ring

Let H be a quasi-Hopf algebra.

Green ring

The Green ring $r(H)$ is the abelian group generated by the isomorphism classes $[M]$ of M in mod$H$ modulo the relations $[M \oplus V] = [M] + [V]$, $M, N \in$ mod$H$. The multiplication is given by tensor product: $[M][V] = [M \otimes V]$, where mod$H$ denotes the category of finite dimensional left modules over H.

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$r(H)$ is an associative ring with $1 = [\Bbbk]$, which is a free abelian group with a $\mathbb{Z}$-basis $\{[V] | V \in \text{Ind}(H)\}$, where Ind$(H)$ is the category of finite dimensional indecomposable H-modules.

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Stable Green ring

Recall that the stable module category $\underline{\mathrm{mod}}H$ is the quotient category of mod$H$ modulo the morphisms factoring through the projective modules. The Green ring of the stable module category $\underline{\mathrm{mod}}H$, denote $r_{st}(H)$, is called stable Green ring of H. Z-basis $\{[V] | V \in \mathrm{Ind}(H)/\mathrm{Ind}_{\mathrm{p}}(H)\}$

The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

- $\mathbb{k}$: an algebraically closed field.
- $q \in \mathbb{k}$, $q \neq \pm 1$.

The quantum enveloping algebra $U_q = U_q(\mathfrak{sl}_2)$ is generated, as an algebra, by E, F, K and K^{-1} subject to

$$\begin{aligned} KEK^{-1} &= q^2 E, & KFK^{-1} &= q^{-2} F, \\ KK^{-1} &= K^{-1}K = 1, & [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

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U_q is a Hopf algebra with Δ, ε and S given by

$$\triangle(K) = K \otimes K, \triangle(E) = E \otimes K + 1 \otimes E, \triangle(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\varepsilon(K) = 1,\ \varepsilon(E) = \varepsilon(F) = 0,$$
$$S(K) = K^{-1},\ S(E) = -EK^{-1},\ S(F) = -KF.$$

Small quantum group $u_q(\mathfrak{sl}_2)$

Suppose that q is a root of unity with $m = |q| > 2$. Put

$$n = \begin{cases} m, & \text{if } m \text{ is odd}, \\ \frac{m}{2}, & \text{if } m \text{ is even}. \end{cases}$$

$$u_q(\mathfrak{sl}_2) := U_q(\mathfrak{sl}_2)/(K^n - 1, E^n, F^n) =: \overline{U}_q.$$

[G. Lusztig, J. Amer. Math. Soc., 1990, 3(1): 257-296].

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When $m = 2n$ is even,

$$\mathfrak{U}_q(\mathfrak{sl}_2) := U_q(\mathfrak{sl}_2)/(K^{2n} - 1, E^n, F^n).$$

[R. Suter, Comm. Math. Phys., 1994, 163(2): 359-393]
[J. Xiao, Can. J. Math., 1997, 49(4): 772-787]
[H. Kondo and Y. Saito, J. Algebra, 2011, 330: 103-129]
[D. Su and S. Yang, J. Math. Phys. 58 (2017), 091704]

Small quasi-quantum $\widetilde{u_q}(\mathfrak{sl}_2)$

Let $n > 2$ and d be two positive odd integers, q a primitive n-th unity, and let I be the ideal of $U_q(\mathfrak{sl}_2)$ generated by E^n, F^n and $K^{dn} - 1$.

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For any $0 \leqslant i \leqslant n^2 - 1$. Let

$$1_i = \frac{1}{n^2} \sum_{r=0}^{n^2-1} \mathbf{q}^{ir} K^r,$$

where a $\mathbf{q}$ is primitive n^2-th root of unity such that $\mathbf{q}^n = q$.

Small quasi-quantum $\widetilde{u}_q(\mathfrak{sl}_2)$

The comultiplication

$$\Delta(K) = K \otimes K,$$

$$\Delta(E) = (E \otimes 1)(T_1 \otimes K^{n^2-n} + T_2 \otimes 1) + K \otimes E,$$

$$\Delta(F) = (F \otimes 1)(T_3 \otimes K^{n^2-1} + T_4 \otimes K^{n-1}) + 1 \otimes F,$$

where

$$T_1 = 1_0 + 1_1 +, \cdots, 1_{2n-1},$$

$$T_2 = 1_{2n} +, \cdots, 1_{n^2-1},$$

$$T_3 = 1_0 + 1_1 +, \cdots, 1_{n^2-2n-1},$$

$$T_4 = 1_{n^2-2n} + 1_{n^2-2n+1}, \cdots, 1_{n^2-1}.$$

Moreover, we have

$$T_1 + T_2 = 1, \;\; T_3 + T_4 = 1.$$

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Liu constructed small quasi-quantum $Qu_q(sl_2)$.

[G. Liu, Math. Res. Lett 21 (2014), 585-603.]

Simple module of H

We consider the finite dimensional representations of H.

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For any integer $1 \leqslant l \leqslant n$, let V_l be a vector space of dimension l with a $\Bbbk$-basis $\{m_1, m_2, \cdots, m_l\}$. Then one can easily check that V_l is an H-module with the action determined by:

$$Em_i = \begin{cases} m_{i+1}, & 1 \leqslant i < l, \\ 0, & i = l, \end{cases} \qquad Fm_i = \begin{cases} 0, & i = 1, \\ \beta_{i-1}(l) m_{i-1}, & 1 < i \leqslant l, \end{cases}$$

$$Km_i = q^{2i-l-1} m_i, \ 1 \leqslant i \leqslant l,$$

where $\beta_i(l) = \frac{(i)_{q^2}(q^{-2i+l+1} - q^{1-l})}{q - q^{-1}}$, $1 \leqslant i < l$.

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Lemma

For any $1 \leqslant l \leqslant n$, V_l is a simple H-module.

Simple module of H

For any $1 \leqslant t \leqslant n-1$ and $r \in \mathbb{Z}$, let $V(t,r)$ be a vector space of dimension n with a $\Bbbk$-basis $\{v_1, v_2, \cdots, v_n\}$. Then one can easily check that $V(t,r)$ is an H-module with the action determined by:

$$Ev_i = \begin{cases} v_{i+1}, & 1 \leqslant i < n, \\ 0, & i = n, \end{cases} \qquad Fv_i = \begin{cases} 0, & i = 1, \\ \gamma_{i-1} v_{i-1}, & 2 < i \leqslant n, \end{cases}$$
$$Kv_i = \mathbf{q}^t q^{2(r+i-1)} v_i, \ 1 \leqslant i \leqslant n.$$

where $\gamma_j = \frac{1}{q-q^{-1}}(\mathbf{q}^{-t}\frac{q^{-2r}-q^{-2(r+j)}}{1-q^{-2}} - \mathbf{q}^t\frac{q^{2r}-q^{2(r+j)}}{1-q^2}) \neq 0$ for $1 \leqslant j \leqslant n-1$.

Simple module of H

Lemma

Let $1 \leqslant t \leqslant n-1$ and $\beta \in \mathbb{Z}$. Then $V(t,\beta)$ is a simple H-module. Moreover, if $\beta \equiv r \pmod{n}$ with $0 \leqslant r \leqslant n-1$, then $V(t,\beta) \cong V(t,r)$.

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Theorem

Let V be a simple H-module. Then V is isomorphic to V_l or $V(t,r)$ for some $1 \leqslant l \leqslant n$, $1 \leqslant t \leqslant n-1$ and $0 \leqslant r \leqslant n-1$.

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Theorem

$\{V_l, V(t,r) | 1 \leqslant l \leqslant n, 1 \leqslant t \leqslant n-1, 0 \leqslant r \leqslant n-1\}$ is a representative set of isomorphic classes of simple H-modules.

Indecomposable modules over H

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$$H \cong He_0 \times He_1 \times \cdots \times He_{n-1}.$$

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Lemma

For $0 \leqslant i \leqslant n-1$, He_i is generated, as an algebra, by $E_i = Ee_i, F_i = Fe_i, K_i = Ke_i, K_i^{-1} = K^{-1}e_i$ subject to following relations

$$K_iE_iK_i^{-1} = q^2E_i, K_iF_iK_i^{-1} = q^{-2}F_i, [E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

$$K_iK_i^{-1} = 1, E_i^n = F_i^n = 0, K_i^n = q^{n-i}e_i$$

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In particular, He_0 is isomorphic,(as algebra) to the small quantum group $\overline{U}_q$.

Proposition

Let M be an H-module. Then $M \in IndH$ if and only if there exists an $0 \leqslant i \leqslant n-1$ such that $M \in IndHe_i$..

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Let M be an H-module. Then M is a simple H-module if and only if there exists an $0 \leqslant i \leqslant n-1$ such that M is a simple He_i-module.

In fact, one can check that

$$\{V_l | 1 \leqslant l \leqslant n\} \rightarrow He_0$$

$$\{V(t,r) | 1 \leqslant t \leqslant n-1, 0 \leqslant r \leqslant n-1\} \rightarrow He_i \text{ with } i \neq 0$$

Indecomposable modules over H

He_i has a $\Bbbk$-basis $\{E_i^j F_i^l K_i^t | 0 \leqslant j, l, t \leqslant n-1\}$. For any $0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$, let

$$T_j^i = \frac{1}{n}\left(\sum_{k=0}^{n-1} \mathbf{q}^{((j-1)n+i)k} K_i^k\right).$$

Then $\{T_j^i | 1 \leqslant j \leqslant n\}$ is a set of orthogonal idempotents of He_i, and $\sum_{j=1}^{n} T_j^i = e_i$.

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Then $\{T_j^i | 1 \leqslant j \leqslant n\}$ is a set of orthogonal idempotents of He_i, and $\sum_{j=1}^{n} T_j^i = e_i$.

Hence, (as He_i-modules) we have

$$He_i = \oplus_{j=1}^{n} He_i T_j^i.$$

One can check

$$K_i T_j^i = \mathbf{q}^{(1-j)n-i} T_j^i \text{ and } K_i^{-1} T_j^i = \mathbf{q}^{i-(1-j)n} T_j^i.$$

Thus, $He_i T_j^i$ has a $\Bbbk$-basis

$$\{E_i^s F_i^l T_j^i | 0 \leqslant s, l \leqslant n-1\}.$$

Indecomposable modules over H

For any $1 \leqslant i \leqslant n-1$, $1 \leqslant j, w \leqslant n$ and $1 \leqslant s \leqslant w$, define $a_{ws}^{ij} \in \mathbb{k}$ by $a_{ww}^{ij} = 1$ for $1 \leqslant w \leqslant n$,

$$a_{w1}^{ij} = \frac{q^{2(w-1)}\mathbf{q}^{(1-j)n-i} - q^{2(1-w)}\mathbf{q}^{i-(1-j)n}}{q - q^{-1}} \neq 0,$$

and $a_{ws}^{ij} = a_{w1}^{ij} + a_{w-1,s-1}^{ij}$ for $2 \leqslant w \leqslant n$ and $2 \leqslant s \leqslant w-1$.

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For any $1 \leqslant w \leqslant n$, we define

$$A_w^{ij} = \sum_{r=0}^{w-1} \prod_{t=0}^{r} a_{w,w-t}^{ij} E_i^{w-1-r} F_i^{n-1-r} \in He_i.$$

Proposition

Let $1 \leqslant i \leqslant n-1$, $1 \leqslant j, w \leqslant n$. Then the He_i-submodule $\langle A_w^{ij} T_j^i \rangle$ of $He_i T_j^i$ generated by $A_w^{ij} T_j^i$ is isomorphic to $V(n-i, r)$.

Note that: $r = \frac{n+\overline{2w-j}}{2}$ for $\overline{2w-j}$ being odd, and $r = \frac{\overline{2w-j}}{2}$ for $\overline{2w-j}$ being even, where $2w-j \equiv \overline{2w-j} \pmod n$, and $0 \leqslant \overline{2w-j} < n$.

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Proposition

Let $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$. Then the sum $\langle A_1^{ij} T_j^i \rangle + \langle A_2^{ij} T_j^i \rangle + \cdots \langle A_n^{ij} T_j^i \rangle$ is direct. Moreover, we have

$$He_i T_j^i \cong \oplus_{r=0}^{n-1} V(n-i, r).$$

Indecomposable modules over H

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Let $1 \leqslant t \leqslant n-1$ and $0 \leqslant r \leqslant n-1$. Then $V(t,r)$ is a projective simple H-module.

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In what follows, we only need to investigate indecomposable He_0-modules.

$$He_0 \cong \overline{U}_q \text{ as algebras}$$

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Simple modules: $V_l,\ 1 \leqslant l \leqslant n$.

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Band modules: $M_s(l, \eta)\ 1 \leqslant l \leqslant n-1,\ s \geqslant 1,\ \eta \in \overline{\mathbb{k}}.$

For the definition of String and Band modules, refer to [Karin. Erdmann, Block of Tame Representation type and Related Algebra]

Indecomposable modules over H

String modules : $\Omega^{\pm m}V_l$, where $m \geqslant 1$, $1 \leqslant l < n$, Ω is the syzygy functor and Ω^{-1} is the cosyzygy functor.

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$$\cdots \to 3P_l \to 2P_{n-l} \to P_l \to V_l \to 0$$

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and the minimal injective resolutions of V_l

$$0 \to V_l \to P_l \to 2P_{n-l} \to 3P_l \to \cdots .$$

Indecomposable modules over H

For any non-projective modules C, we have exact sequence

$$0 \to DTr(C) \to B \to C \to 0.$$

For the definition of $DTr(C)$, refer to
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Theorem

$\{V_l, V(t,r) | 1 \leqslant l \leqslant n, 1 \leqslant t \leqslant n-1, 0 \leqslant r \leqslant n-1\} \cup \{P_l, \Omega^{\pm s} V_l, M_s(l,\eta) | 1 \leqslant l \leqslant n-1, s \geqslant 1, \eta \in \overline{\mathbb{k}}\}$ is a representative set of isomorphic classes of indecomposable H-modules.

Let us return to $\overline{U}_q$

Tensor Products of Indecomposable H-modules

Let us return to $\overline{U}_q$

Now let $E' = F$, $F' = E$, $K' = K^{-1}$ in $\overline{U}_q$. Then, one can describe $\overline{U}_q$ as follows. The small quantum group $\overline{U}_q$ is generated, as an algebra, by E', F' and K' subject to following relations:

$$K'E'K'^{-1} = q^2E',\ K'F'K'^{-1} = q^{-2}F',$$

$$[E', F'] = \frac{K' - K'^{-1}}{q - q^{-1}},\ K'K'^{-1} = K'^{-1}K' = 1.$$

$$E'^n = F'^n = 0, K'^n = 1$$

The coalgebra structure is given by

$$\Delta(K') = K' \otimes K',$$

$$\Delta(E') = E' \otimes 1 + K' \otimes E',$$

$$\Delta(F') = F' \otimes K'^{-1} + 1 \otimes F'.$$

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The comultiplication of small quasi-quantum group H (He_0):

$$\Delta(E) = (E \otimes 1)(T_1 \otimes K^{n^2-n} + T_2 \otimes 1) + K \otimes E,$$
$$\Delta(F) = (F \otimes 1)(T_3 \otimes K^{n^2-1} + T_4 \otimes K^{n-1}) + 1 \otimes F.$$

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Let $M, N \in IndHe_0$. Then we have

$$E(M \otimes N) = (E \otimes 1 + K \otimes E)(M \otimes N),$$

$$F(M \otimes N) = (F \otimes K^{-1} + 1 \otimes F)(M \otimes N).$$

Tensor Products of Indecomposable H-modules

Remark

The stable Green ring $r_{st}(\overline{U}_q)$ of small quantum group $\overline{U}_q$ is isomorphic to stable Green ring $r_{st}(\widetilde{U}_q)$ of small quasi-quantum group $\widetilde{U}_q$.

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Tensor Products of Indecomposable H-modules

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We only need to consider the following tensor products:
(1) Tensor products of simple He_i-modules with indecomposable He_0-modules,$1 \leqslant i \leqslant n-1$.
(2) Tensor products of simple He_i-modules with simple He_j-modules for $1 \leqslant i, j \leqslant n-1$.

Proposition

Let $1 \leqslant l \leqslant n$, $1 \leqslant t \leqslant n-1$ *and* $0 \leqslant r \leqslant n-1$.

(1)*If* l *is odd, then* $V_l \otimes V(t,r) \cong V(t,r) \otimes V_l \cong \oplus_{j=0}^{l-1} V(t, r - \frac{l-1}{2} + j)$

(2)*If* l *is even, then* $V_l \otimes V(t,r) \cong V(t,r) \otimes V_l \cong \oplus_{j=0}^{l-1} V(t, r + \frac{n-l+1}{2} + j)$

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Let $1 \leqslant l, t \leqslant n-1$ *and* $0 \leqslant r \leqslant n-1$. *Then*

$$P_l \otimes V(t,r) \cong V(t,r) \otimes P_l \cong 2 \oplus_{j=0}^{n-1} V(t,j)$$

Tensor Products of Indecomposable H-modules

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Let $s \geqslant 1$, $1 \leqslant t, l \leqslant n-1$, *and* $0 \leqslant r \leqslant n-1$. *Then*

Tensor Products of Indecomposable H-modules

Proposition

Let $s \geqslant 1$, $1 \leqslant t, l \leqslant n-1$, and $0 \leqslant r \leqslant n-1$. Then

(1) If s and l are odd, then

$$\begin{aligned} & \Omega^{\pm s} V_l \otimes V(t,r) \cong V(t,r) \otimes \Omega^{\pm s} V_l \\ \cong \quad & (s-l+1) \oplus_{j=0}^{n-1} V(t,j) \oplus (\oplus_{0 \leqslant j \leqslant n-1, j \neq r} \oplus_{j'=0}^{l-1} V(t, j - \tfrac{l-1}{2} + j')) \end{aligned}$$

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Proposition

Let $s \geqslant 1$, $1 \leqslant l, t \leqslant n-1$, $0 \leqslant r \leqslant n-1$ and $\eta \in \overline{\Bbbk}$. Then

$$M_s(l,\eta) \otimes V(t,r) \cong V(t,r) \otimes M_s(l,\eta) \cong s \oplus_{j=0}^{n-1} V(t,j)$$

Tensor Products of Indecomposable H-modules

Proposition

Let $1 \leqslant t, t' \leqslant n-1$ *with* $t+t' \neq n$, *and* $0 \leqslant r, r' \leqslant n-1$.

(1) *If* $t+t' < n$, *then* $V(t,r) \otimes V(t',r') \cong \oplus_{j=0}^{n-1} V(t+t', j)$

(2) *If* $t+t' > n$, *then* $V(t,r) \otimes V(t',r') \cong \oplus_{j=0}^{n-1} V(t+t'-n, j)$

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Theorem

Let $1 \leqslant t, t' \leqslant n-1$ *with* $t+t' = n$, $0 \leqslant r, r' \leqslant n-1$, *and let* $r+r' \equiv u \pmod{n}$.

(1) *If* $0 \leqslant u \leqslant \frac{n+1}{2}$, *then*

$$V(t,r) \otimes V(t',r') \cong V_n \oplus (\oplus_{u \leqslant j \leqslant \frac{n-1}{2}} P_{2j}) \oplus (\oplus_{1 \leqslant j \leqslant u-1} P_{n-2(u-j)})$$

(2) *If* $u > \frac{n+1}{2}$, *then*

$$V(t,r) \otimes V(t',r') \cong V_n \oplus (\oplus_{n+1-u \leqslant j \leqslant \frac{n-1}{2}} P_{2j}) \oplus (\oplus_{1 \leqslant j \leqslant n-u} P_{n-2(n+1-u-j)})$$

Next work

1. Describe the Green ring, stable Green ring.

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2. Consider the representations of $\widetilde{u}_q(\mathfrak{sl}_2)$ in case $d \neq n$.

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Thank You For Your Attention!