Renormalization of weak quasisymmetric functions

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(Joint work with Li Guo and Bin Zhang)

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Outline

- Motivation
- Weak quasisymmetric functions
- Renormalization of weak quasisymmetric functions

Motivation

$$\sum_{i=1}^{\infty} 1 = 1 + 1 + 1 + \cdots$$

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- A **weak composition** ($\mathcal{WC}$) is a vector $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{N}^k$.

  A **left weak composition** ($\mathcal{LWC}$) is a weak composition $(\alpha_1, \cdots, \alpha_k)$ with $\alpha_k > 0$.

  A **composition** ($\mathcal{C}$) is a weak composition $(\alpha_1, \cdots, \alpha_k)$ with $\alpha_i > 0$, $i = 1, 2, \cdots, k$.

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- A quasisymmetric function (QSym) in the variables $X = (x_1, x_2, \cdots)$ is a bounded degree formal power series $f(X) \in \mathbb{Q}[[X]]$ such that for any composition $(\alpha_1, \alpha_2, \cdots, \alpha_k)$, we have $a = b$ in

$$f(x_1, x_2, \cdots) = ax_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + \cdots + bx_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} + \cdots .$$

  whenever $i_1 < i_2 < \cdots < i_k$.

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$$f(x_1, x_2, \cdots) = a x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + \cdots + b x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} + \cdots .$$

  whenever $i_1 < i_2 < \cdots < i_k$.

- Example $M_{(2,1)} = \sum_{i_1 < i_2} x_{i_1}^2 x_{i_2} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots + x_{i_1}^2 x_{i_2} + \cdots .$

QSym: Monomial basis M_α

- Fix a composition $\alpha = (\alpha_1, \alpha_2 \cdots, \alpha_k)$, the monomial quasisymmetric function indexed by α is
$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$
Define $\mathbf{QSym}_n = \mathrm{Span}\{M_\alpha | \alpha \models n\}$ and $\mathbf{QSym} = \bigoplus_{n \geq 0} \mathbf{QSym}_n$.

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- **Question**: Why and How to deal with this divergence?

Why need to deal with this divergence?

- Rota-Baxter algebra

Rota's Conjecture/Question In 1995, Rota conjectured that Baxter algebras represent the ultimate and most natural generalization of the algebra of symmetric functions.

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Rota Program Study generalizations of symmetric functions in the context of Rota-Baxter algebras.

Y. Li, On weak peak quasisymmetric functions, *J. Combin. Theory, Ser. A* **158** (2018), 449-491.

L. Guo, J.-Y. Thibon and H. Yu, Weak quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras, *Adv. Math.* **344** (2019) 1–34.

- Free unitary Rota-Baxter algebra
- $(Ш(x), P_x)$ is the free commutative unitary Rota-Baxter algebra on x, where

$$Ш(x) := \bigoplus_{k\geq 0}\left(\mathbf{k}[x]^{\otimes(k+1)}\right) = \bigoplus_{\alpha_i\geq 0, 0\leq i\leq k} \mathbf{k}x^{\alpha_0}\otimes\cdots\otimes x^{\alpha_k}$$

and $P_x : x^{\alpha_0}\otimes x^{\alpha_1}\otimes\cdots\otimes x^{\alpha_k}\mapsto 1\otimes x^{\alpha_0}\otimes x^{\alpha_1}\otimes\cdots\otimes x^{\alpha_k}$.

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and $P_x : x^{\alpha_0}\otimes x^{\alpha_1}\otimes\cdots\otimes x^{\alpha_k}\mapsto 1\otimes x^{\alpha_0}\otimes x^{\alpha_1}\otimes\cdots\otimes x^{\alpha_k}$.

- The pure tensor $1\otimes x^{\alpha_0}\otimes x^{\alpha_1}\otimes\cdots\otimes x^{\alpha_k}$ can be viewed as

$$M_{(\alpha_0,\alpha_1,\cdots,\alpha_k)} = \sum_{i_0<i_1<\cdots<i_k} x_{i_0}^{\alpha_0}x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k},$$

where $(\alpha_0,\alpha_1,\cdots,\alpha_k)$ is a weak composition.

Why need to deal with this divergence?

- By specializing x_i to $1/i$, M_α gives the multiple zeta value (MZV)

$$\zeta(\alpha_1, \alpha_2, \cdots, \alpha_k) := \sum_{0<i_1<i_2<\cdots<i_k} \frac{1}{i_1^{\alpha_1} i_2^{\alpha_2} \cdots i_k^{\alpha_k}},$$

which is convergent for $\alpha_k \geq 2, \alpha_j \geq 1, 1 \leq j \leq k-1$.

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Divergent MZVs have been defined through various renormalization processes. The success suggests that M_α can be similarly treated, providing a testing ground in applying the renormalization method to divergencies in mathematics.

L. Guo and B. Zhang, Renormalization of multiple zeta values, *J. Algebra*, **319** (2008), 3770–3809.

K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, *Compos. Math.*, **142** (2006), 307–338.

D. Manchon and S. Paycha, Nested sums of symbols and renormalized multiple zeta values, *Int. Math. Res. Not. IMRN*, **2010** (2010), 4628–4697.

Why need to deal with this divergence?

- Power series realizations in abstractly combinatorial Hopf algebras.

  Bijections $\mathbb{QC} \to$ **QSym** and $\mathbb{Q}\mathcal{LWC} \to$ **LWQSym**, defined by

  $$\alpha = (\alpha_1, \alpha_2 \cdots, \alpha_k) \mapsto M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k},$$

  give a power series realization of the quasi-shuffle algebras.

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- What is the power series realization of $\mathbb{Q}\mathcal{WC}$?

L. Foissy, J.-C. Novelli and J.-Y. Thibon, Polynomial realizations of some combinatorial Hopf algebras, *J. Noncommut. Geom.*, **8** (2014), 141–162.

L. Guo, H. Yu and J. Zhao, Rota–Baxter algebras and left weak quasi-symmetric functions, *Ramanujan J.* **44** (2017) 567–596.

R. Maurice, A polynomial realization of the Hopf algebra of uniform block permutations, *Adv. Appl. Math.*, **51** (2013), 285–308.

Y. Zhang, X. Gao, Hopf algebras of planar binary trees: an operated algebra approach. *J. Algebraic Combin.* **51** (2020),567–588.

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Formal regularization (Weak quasi-symmetric functions)

- Let $\widetilde{\mathbb{N}} := \mathbb{N} \cup \{\varepsilon\}$, satisfying $0+\varepsilon=\varepsilon+0=\varepsilon+\varepsilon=\varepsilon$ and $n+\varepsilon=\varepsilon+n=n$ for all $n \geq 1$.

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- Each $\widetilde{\mathbb{N}}$-composition $\alpha = (\alpha_1, \alpha_2 \cdots, \alpha_k)$ corresponds to a **monomial weak quasisymmetric function**
$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$
For example,
$$M_0 \rightarrow M_\varepsilon = \sum_i x_i^\varepsilon = x_1^\varepsilon + x_2^\varepsilon + \cdots,$$
$$M_{(3,0)} \rightarrow M_{(3,\varepsilon)} = \sum_{i_1 < i_2} x_{i_1}^3 x_{i_2}^\varepsilon = x_1^3 x_2^\varepsilon + x_1^3 x_3^\varepsilon + \cdots + x_2^3 x_3^\varepsilon + x_2^3 x_4^\varepsilon + \cdots.$$

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- **RQSym** $= \mathrm{span}\{M_\alpha | \alpha \in \widetilde{\mathcal{C}}\}$ is called the algebra of **weak quasi-symmetric functions**.

Formal regularization (Weak quasi-symmetric functions)

Theorem (L. Guo, H. Y. & J. Zhao, 2017)

$\mathbf{k}[x] \otimes$ **LWQSym** is the free commutative nonunitary Rota-Baxter algebra of weight 1 on x.

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Theorem (L. Guo, Y. Thibon & H. Y., 2019)

(1) $\mathbf{k}[x] \otimes$ **RQSym** is the free commutative unitary Rota-Baxter algebra of weight 1 on x;

(2) **QSym** is a Hopf subalgebra and Hopf quotient algebra of **RQSym**.

L. Guo, H. Yu and J. Zhao, Rota–Baxter algebras and left weak composition quasi-symmetric functions, *Ramanujan J.* **44** (2017) 567–596.

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- In order to extract finite values from these integrals, physicists developed a procedure called renormalization, which was remarkably successful as a physical theory, but weak in math foundation.
- Connes and Kreimer provided a mathematical framework for renormalization.

A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.*, **210** (2000), 249–273.

D. Manchon, Hopf algebras in renormalisation, *Handbook of Algebra* (M. Hazewinkel ed.) **5** (2008), 365-427, arXiv:math. QA/0408405.

Renormalization in quantum field theory

- A **connected filtered Hopf algebra** is a Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$ satisfying

$$H^{(0)} = \mathbf{k}, \quad H^{(n)} \subseteq H^{(n+1)}, \quad H^{(p)}H^{(q)} \subseteq H^{(p+q)},$$
$$\Delta(H^{(n)}) \subseteq \sum_{p+q=n} H^{(p)} \otimes H^{(q)}, \quad S(H^{(n)}) \subseteq H^{(n)}.$$

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- Fix λ in a base field $\mathbf{k}$. A **Rota-Baxter algebra** of weight λ is a pair (R, P) consisting of an algebra R and a linear operator $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall\, x, y \in R.$$

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- Laurent series A Rota-Baxter algebra of weight -1:

$$R = \mathbb{C}[t^{-1}, t]] = \left\{ \sum_{n=k}^{\infty} a_n t^n | a_n \in \mathbb{C}, k \leq n < \infty, k \in \mathbb{Z} \right\}, P\left(\sum_{n=k}^{\infty} a_n t^n \right) = \sum_{n=k}^{-1} a_n t^n.$$

Algebraic Birkhoff Factorization

Algebraic Birkhoff Factorization Theorem For a given triple (H, R, ϕ) consisting of

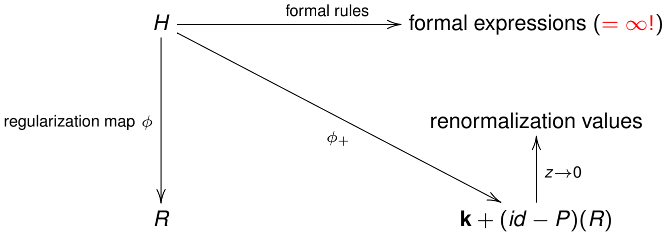
- a connected filtered Hopf algebra H,
- a commutative Rota-Baxter algebra R on which the Rota-Baxter operator $P : R \to R$ is idempotent, and
- an algebra homomorphism $\phi : H \to R$ serving as the regularization map,

there is a unique algebra homomorphisms decomposition

$$\phi = \phi_+ \star \phi_-^{\star(-1)} \quad \text{where} \quad \begin{cases} \phi_- : H & \to \mathbf{k} + P(R) \quad \text{(counter term)} \\ \phi_+ : H & \to \mathbf{k} + (\mathrm{id} - P)(R) \quad \text{(renormalization).} \end{cases}$$

Here $\star$ is the convolution product and ϕ_+ is called the renormalization of ϕ.

Algebraic Birkhoff Factorization



Renormalization process

- Quasi-shuffle Hopf algebra of directional weak compositions

$$H_{\mathrm{DWC}} = \mathrm{QS}(\mathbf{k}(\mathbb{N} \times \mathbb{P}))$$

with a canonical basis

$$\mathrm{DWC} := \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k \end{bmatrix} \;\middle|\; \alpha \in \mathbb{N}^k, \beta \in \mathbb{P}^k, k \in \mathbb{N} \right\}.$$

whose elements will be called the **directional weak compositions**.

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- Rota-Baxter algebra $R = \mathbf{LWQSym}[t][z, z^{-1}]]$, where t is a a variable.

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- Regularization map ϕ
For $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] = \left[\begin{smallmatrix} \alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k \end{smallmatrix} \right] \in \mathrm{DWC}$, real number $t \geq 0$ and complex number z with real part $\mathrm{Re}(z) < 0$, we define

$$\phi\left(\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]\right) := \phi\left(\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]\right)(t, z) := \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} e^{(i_1+t)\beta_1 z} \cdots e^{(i_k+t)\beta_k z},$$

which we call the **regularization of the formal expression M_α in the direction** β.

Renormalization process

Theorem (L. Guo, H.Y. & B. Zhang, 2021) The assignment ϕ defines an algebra homomorphism

$$\phi : H_{\mathrm{DWC}} \to \textbf{LWQSym}[t][z^{-1}, z]].$$

If $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is in DWC and α is a left weak composition, then $\phi\left(\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right)$ is an element of **LWQSym**$[t][[z]]$.

L. Guo, H. Yu and B. Zhang, Renormalization of quasisymmetric functions, arXiv:2012.11872.

Renormalization process

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$$Z\left(\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right) = \phi_+\left(\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right)\Big|_{z=0} \in \mathbf{LWQSym}[t]$$

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- Example

$$M_0=-t-\frac{1}{2},\quad M_{(0,0)}=\frac{1}{2}t^2+t+\frac{3}{8},\quad M_{(0,0,0)}=-\frac{1}{6}t^3-\frac{3}{4}t^2-\frac{23}{24}t-\frac{5}{16}.$$

Renormalization process

- For any positive integer k, we have

$$M_{0^k} = \frac{\prod_{i=0}^{k-1}(M_0 - i)}{k!} = \frac{(-1)^k}{k!}\prod_{i=1}^{k}\left(t + i - \frac{1}{2}\right).$$

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- Let $\alpha = (\alpha_1, \ldots, \alpha_j, 0^{k_\alpha})$ be a weak composition, where $\alpha_j \in \mathbb{P}$ and $k_\alpha \in \mathbb{N}$. Denote $\alpha' = (\alpha_1, \ldots, \alpha_{j-1})$. Then

$$M_\alpha = \sum_{p=0}^{k_\alpha} \frac{(-1)^p \prod_{i=1}^{k_\alpha - p}(M_0 - \ell(\alpha) + i)}{(k_\alpha - p)!} M_{(\alpha' \sqcup\!\sqcup\, 0^p, \alpha_j)}$$

$$= (-1)^{k_\alpha} \sum_{p=0}^{k_\alpha} \frac{\prod_{i=1}^{k_\alpha - p}(t + \ell(\alpha) - i + \frac{1}{2})}{(k_\alpha - p)!} M_{(\alpha' \sqcup\!\sqcup\, 0^p, \alpha_j)}.$$

Renormalization of weak quasisymmetric functions

Let

$$\textbf{RenQSym} = \mathrm{span}\{M_\alpha | \alpha \in \mathcal{WC}\}$$

whose elements will be called renormalized quasisymmetric functions.

Theorem (L. Guo, H.Y. & B. Zhang, 2021)

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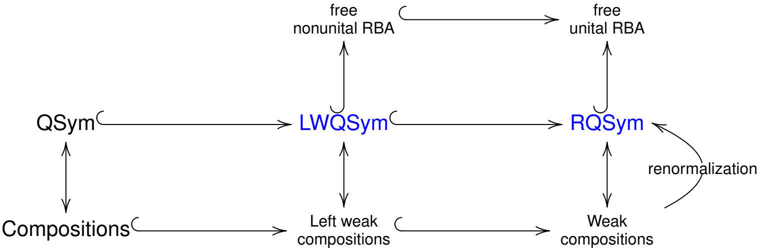

Thank you !