BiHom-Novikov algebras and perturbations of BiHom-Novikov-Possion algebras

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- Preliminaries

BiHom-associative algebra

Definition

A BiHom-associative algebra is a 4-tuple (A, μ, α, β)*, where* A *is a linear space and* $\alpha, \beta : A \to A$ *and* $\mu : A \otimes A \to A$ *are linear maps such that* $\alpha \circ \beta = \beta \circ \alpha$, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, $\beta(x \cdot y) = \beta(x) \cdot \beta(y)$ *and*

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z), \tag{1}$$

G. Graziani, A. Makhlouf, C. Menini, F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, Symmetry Integrability Geom. Methods Appl. 11 (2015), 086 (34 pages).

If $\alpha = \beta$, then a BiHom-associative algebra is a Hom-associative algebra.

MSC2020-Mathematical Sciences Classication System: 17D30 (non-Lie) Hom algebras and topics.

If $\beta = \alpha^{-1}$, then a BiHom-associative algebra is a monoidal Hom-associative algebra.
S. Caenepeel, I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011), 2216–2240.

Yau twist: If A is an associative algebra and $\alpha, \beta : A \to A$ are algebra maps, then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra, here $a * b = \alpha(a)\beta(b)$.

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- BiHom-Novikov algebras and BiHom-Novikov-Poisson algebras

L. Liu, A. Makhlouf, C. Menini, F. Panaite, *BiHom-Novikov algebras and infinitesimal BiHom-bialgebras*, J. Algebra 560 (2020), 1146–1172.

L. Liu, A. Makhlouf, C. Menini, F. Panaite, *Tensor prosucts and pertubations of BiHom-Novikov-poisson algebras*, J. Geometry and Physics 161 (2021), 104026.

- BiHom-Novikov algebras and BiHom-Novikov-Poisson algebras

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Definition 2.1

A BiHom-Novikov algebra is a 4-tuple (A, μ, α, β), where A is a linear space, $\mu : A \otimes A \to A$ is a linear map and $\alpha, \beta : A \to A$ are commuting linear maps (called the structure maps of A), satisfying the following conditions, for all $x, y, z \in A$:

$$\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \beta(x \cdot y) = \beta(x) \cdot \beta(y), \tag{2}$$

$$(\beta(x) \cdot \alpha(y)) \cdot \beta(z) - \alpha\beta(x) \cdot (\alpha(y) \cdot z)$$
$$= (\beta(y) \cdot \alpha(x)) \cdot \beta(z) - \alpha\beta(y) \cdot (\alpha(x) \cdot z), \tag{3}$$

$$(x \cdot \beta(y)) \cdot \alpha\beta(z) = (x \cdot \beta(z)) \cdot \alpha\beta(y). \tag{4}$$

In other words, a BiHom-Novikov algebra is a left BiHom-pre-Lie algebra satisfying (4).

Gel'fand-Dorfman theorem

Theorem 2.2

Let (A,μ,α,β) be a BiHom-commutative algebra. Let $\gamma,\lambda,\xi : A\to A$ be linear maps such that $\gamma(x\cdot y)=\gamma(x)\cdot\gamma(y)$, $\lambda(x\cdot y)=\lambda(x)\cdot\lambda(y)$ and $\xi(x\cdot y)=\xi(x)\cdot\xi(y)$, for all $x,y\in A$. Let $D:A\to A$ be a linear map, assume that any two of the maps $\alpha,\beta,\gamma,\lambda,\xi,D$ commute and the following condition is satisfied:

$$D(a\cdot b)=\gamma(a)\cdot D(b)+D(a)\cdot\gamma(b),\quad \forall\ a,b\in A. \tag{5}$$

Define a new multiplication on A by $a*b=\lambda(a)\cdot\xi D(b)$, for all $a,b\in A$. Then $(A,*,\lambda\alpha,\xi\beta\gamma)$ is a BiHom-Novikov algebra.

Definition 2.3 A BiHom-associative algebra (A,μ,α,β) is called BiHom-commutative if

$$\beta(a)\cdot\alpha(b)=\beta(b)\cdot\alpha(a),\quad \forall\ a,b\in A. \tag{6}$$

Take in the previous theorem $\lambda = \alpha^p$, $\gamma = \beta^r$, $\xi = id_A$.

Corollary 2.4 Let (A, μ, α, β) be a BiHom-commutative algebra. Let p and r be some natural numbers and let $D : A \to A$ be a linear map commuting with α and β and satisfying the condition

$$D(ab) = \beta^r(a) \cdot D(b) + D(a) \cdot \beta^r(b), \quad \forall \; a, b \in A.$$

Define a new multiplication on A by $a * b = \alpha^p(a) \cdot D(b)$, for all $a, b \in A$. Then $\left(A, *, \alpha^{p+1}, \beta^{r+1}\right)$ is a BiHom-Novikov algebra.

In particular, by taking $p = r = 0$, we obtain:

Corollary 2.5 Let (A, μ, α, β) be a BiHom-commutative algebra and let $D : A \to A$ be a derivation in the usual sense (i.e. $D(x \cdot y) = x \cdot D(y) + D(x) \cdot y$, for all $x, y \in A$) commuting with α and β. Then $(A, *, \alpha, \beta)$ is a BiHom-Novikov algebra, where $a * b = a \cdot D(b)$, for all $a, b \in A$.

Take in the previous theorem $\lambda = \alpha^p$, $\gamma = \beta^r$, $\xi = id_A$.

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Corollary 2.6 Let (A, μ) be an associative and commutative algebra, let $\alpha, \beta : A \to A$ be two commuting algebra morphisms and let $D : A \to A$ be a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. Then $(A, *, \alpha, \beta)$ is a BiHom-Novikov algebra, where $a * b = \alpha(a) \cdot D(\beta(b))$, for all $a, b \in A$.

Definition 2.7

A BiHom-Novikov-Poisson algebra is a 5-tuple $(A, \cdot, *, \alpha, \beta)$ such that:
(1) $(A, \cdot, \alpha, \beta)$ is a BiHom-commutative algebra;
(2) $(A, *, \alpha, \beta)$ is a BiHom-Novikov algebra;
(3) the following compatibility conditions hold for all $x, y, z \in A$:

$$
\begin{aligned}
& (\beta(x) * \alpha(y)) \cdot \beta(z) - \alpha\beta(x) * (\alpha(y) \cdot z) \\
= \; & (\beta(y) * \alpha(x)) \cdot \beta(z) - \alpha\beta(y) * (\alpha(x) \cdot z), \qquad (7) \\
& (x \cdot \beta(y)) * \alpha\beta(z) = (x * \beta(z)) \cdot \alpha\beta(y), \qquad (8) \\
& \alpha(x) \cdot (y * z) = (x \cdot y) * \beta(z). \qquad (9)
\end{aligned}
$$

The maps α and β (in this order) are called the structure maps of A.

Proposition 2.8

Under the hypotheses of Corollary 2.5, $(A, \mu, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra.

Corollary 2.9

Let (A, μ) be a commutative and associative algebra, let $\alpha, \beta : A \to A$ be two commuting algebra morphisms, and let $D : A \to A$ be a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. Then $(A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where

$$x \bullet y = \mu(\alpha(x) \otimes \beta(y)), \qquad x * y = \mu(\alpha(x) \otimes D(\beta(y))).$$

Proposition 2.8

Under the hypotheses of Corollary 2.5, $(A, \mu, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra.

Corollary 2.9

Let (A, μ) be a commutative and associative algebra, let $\alpha, \beta : A \to A$ be two commuting algebra morphisms, and let $D : A \to A$ be a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. Then $(A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where

$$x \bullet y = \mu(\alpha(x) \otimes \beta(y)), \qquad x * y = \mu(\alpha(x) \otimes D(\beta(y))).$$

Proposition 2.10 Let $(A, \cdot, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and let $\tilde{\alpha}, \tilde{\beta} : A \to A$ be two morphisms of BiHom-Novikov-Poisson algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute. Then

$$A_{(\tilde{\alpha},\tilde{\beta})} := (A, \tilde{\cdot} := \cdot \circ (\tilde{\alpha} \otimes \tilde{\beta}), \tilde{*} := * \circ (\tilde{\alpha} \otimes \tilde{\beta}), \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$$

is also a BiHom-Novikov-Poisson algebra.

Corollary 2.11 Let $(A, \cdot, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra. Then

$$A^n := (A, \cdot \circ (\alpha^n \otimes \beta^n), * \circ (\alpha^n \otimes \beta^n), \alpha^{n+1}, \beta^{n+1})$$

is also a BiHom-Novikov-Poisson algebra for any $n \geq 0$.

Proposition 2.10 Let $(A, \cdot, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and let $\tilde{\alpha}, \tilde{\beta} : A \to A$ be two morphisms of BiHom-Novikov-Poisson algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute. Then

$$A_{(\tilde{\alpha},\tilde{\beta})} := (A, \tilde{\cdot} := \cdot \circ (\tilde{\alpha} \otimes \tilde{\beta}), \tilde{*} := * \circ (\tilde{\alpha} \otimes \tilde{\beta}), \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$$

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Corollary 2.11 Let $(A, \cdot, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra. Then

$$A^n := (A, \cdot \circ (\alpha^n \otimes \beta^n), * \circ (\alpha^n \otimes \beta^n), \alpha^{n+1}, \beta^{n+1})$$

is also a BiHom-Novikov-Poisson algebra for any $n \geq 0$.

The following result is the special case of Corollary 2.11 with $* = 0$.

Corollary 2.12 Let $(A, \cdot, \alpha, \beta)$ be a BiHom-commutative algebra. Then

$$A^n := (A, \cdot \circ (\alpha^n \otimes \beta^n), \alpha^{n+1}, \beta^{n+1})$$

is also a BiHom-commutative algebra for any $n \geq 0$.

The following result is the special case of Corollary 2.11 with $\cdot = 0$.

Corollary 2.13 Let $(A, *, \alpha, \beta)$ be a BiHom-Novikov algebra. Then

$$A^n := (A, * \circ (\alpha^n \otimes \beta^n), \alpha^{n+1}, \beta^{n+1})$$

is also a BiHom-Novikov algebra for any $n \geq 0$.

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is also a BiHom-Novikov algebra for any $n \geq 0$.

- Perturbations of BiHom-Novikov-Poisson algebras

Lemma 3.1

Let (A, μ, α, β) be a BiHom-associative algebra and let $a \in A$ satisfying

$$\alpha^2(a) = \beta^2(a) = a. \tag{10}$$

Define a new operation on A by

$$\diamond : A \otimes A \to A, \quad x \diamond y = \alpha(x)(\alpha(a)y), \tag{11}$$

for all $x, y \in A$. Then $A' = (A, \diamond, \alpha^2, \beta^2)$ is also a BiHom-associative algebra. If moreover (A, μ, α, β) is BiHom-commutative, then $A' = (A, \diamond, \alpha^2, \beta^2)$ is also BiHom-commutative.

The Hom-associative case of Lemma 3.1:
Let (A, μ, α) be a commutative Hom-associative algebra, i.e. $\alpha(x)(yz) = (xy)\alpha(z)$ and $xy = yx$, for all $x, y, z \in A$. Assume that α is multiplicative with respect to μ and let $a \in A$ such that $\alpha^2(a) = a$. Then obviously (A, μ, α, α) is also a BiHom-commutative algebra and we are in the hypotheses of Lemma 3.1; in this particular case of Lemma 3.1, the multiplication (11) may be rewritten as:

$$\begin{aligned} x \diamond y &= \alpha(x)(\alpha(a)y) = (x\alpha(a))\alpha(y) \\ &= (\alpha(a)x)\alpha(y) = \alpha^2(a)(xy) = a(xy). \end{aligned}$$

What we obtained, $x \diamond y = a(xy)$, is exactly formula (4.1.1) in [D. Yau, *A twisted generalization of Novikov-Poisson algebras*, arXiv:math.RA/1010.3410].
Note that formula (4.1.1) gives a Hom-associative multiplication only in the commutative case; for the noncommutative case, it is clear that the proper formula is (11).

Theorem 3.2

Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ with $\alpha^2(a) = \beta^2(a) = a$. Then $A' = (A, \diamond, *_{\alpha,\beta}, \alpha^2, \beta^2)$ is also a BiHom-Novikov-Poisson algebra, where

$$x \diamond y = \alpha(x)(\alpha(a)y), \quad x *_{\alpha,\beta} y = \alpha(x) * \beta(y), \quad \forall\, x, y \in A.$$

Theorem 3.3

Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ with $\alpha^2(a) = \beta^2(a) = a$. Then $\overline{A} = (A, \cdot_{\alpha,\beta}, \times, \alpha^2, \beta^2)$ is also a BiHom-Novikov-Poisson algebra, where

$$x \cdot_{\alpha,\beta} y = \alpha(x)\beta(y), \quad x \times y = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y).$$

Theorem 3.2

Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ with $\alpha^2(a) = \beta^2(a) = a$. Then $A' = (A, \diamond, *_{\alpha,\beta}, \alpha^2, \beta^2)$ is also a BiHom-Novikov-Poisson algebra, where

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$$x \cdot_{\alpha,\beta} y = \alpha(x)\beta(y), \quad x \times y = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y).$$

Forgetting about the BiHom-associative product $\cdot_{\alpha,\beta}$ in Theorem 3.3, we obtain a non-trivial way to construct a BiHom-Novikov algebra from a BiHom-Novikov-Poisson algebra:

Corollary 3.4 Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then $(A, \times, \alpha^2, \beta^2)$ is a BiHom-Novikov algebra, where $x \times y = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y)$, for all $x, y \in A$.

Corollary 3.5 Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a, b \in A$ elements such that $\alpha^2(a) = \beta^2(a) = a$ and $\alpha^4(b) = \beta^4(b) = b$. Then $\widetilde{A} = (A, \diamond, \boxtimes, \alpha^4, \beta^4)$ is also a BiHom-Novikov-Poisson algebra, where

$$\begin{aligned} x \diamond y &= \alpha^3(x)(\alpha^3\beta(b)\beta^2(y)), \quad \forall\, x, y \in A, \\ x \boxtimes y &= \alpha^3(x) * \beta^3(y) + \alpha^3(x)(\alpha(a)\beta^2(y)), \quad \forall\, x, y \in A. \end{aligned}$$

Forgetting about the BiHom-associative product $\cdot_{\alpha,\beta}$ in Theorem 3.3, we obtain a non-trivial way to construct a BiHom-Novikov algebra from a BiHom-Novikov-Poisson algebra:

Corollary 3.4 Let $(A,\mu,*,\alpha,\beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a)=\beta^2(a)=a$. Then $(A,\times,\alpha^2,\beta^2)$ is a BiHom-Novikov algebra, where $x\times y = \alpha(x)*\beta(y)+\alpha(x)(\alpha(a)y)$, for all $x,y\in A$.

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$$\begin{aligned} x\diamond y &= \alpha^3(x)(\alpha^3\beta(b)\beta^2(y)), \quad \forall\, x,y\in A,\\ x\boxtimes y &= \alpha^3(x)*\beta^3(y)+\alpha^3(x)(\alpha(a)\beta^2(y)), \quad \forall\, x,y\in A. \end{aligned}$$

Corollary 3.6 Let (A, μ) be a commutative and associative algebra, $\alpha, \beta : A \to A$ two commuting algebra morphisms, and $D : A \to A$ a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. Let $a, b \in A$ be elements such that $\alpha^2(a) = \beta^2(a) = a$ and $\alpha^4(b) = \beta^4(b) = b$. Then $(A, \Diamond, \boxdot, \alpha^4, \beta^4)$ is a BiHom-Novikov-Poisson algebra, where

$$\begin{aligned} x \Diamond y &= \alpha^4(x)\beta^2(b)\beta^4(y), \quad \forall\, x, y \in A, \\ x \boxdot y &= \alpha^4(x)D(\beta^4(y)) + \alpha^4(x)\beta(a)\beta^4(y), \quad \forall\, x, y \in A. \end{aligned}$$

- From BiHom-Novikov-Poisson algebras to BiHom-Poisson algebras

Definition 4.1

A BiHom-Lie algebra $(L, [\cdot,\cdot], \alpha, \beta)$ is a 4-tuple in which L is a linear space, $\alpha, \beta : L \to L$ are linear maps and $[\cdot,\cdot] : L \times L \to L$ is a bilinear map, such that

$$\alpha \circ \beta = \beta \circ \alpha, \tag{12}$$

$$\alpha([x,y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x,y]) = [\beta(x), \beta(y)], \tag{13}$$

$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \qquad \text{(BiHom-skew-symmetry)} \tag{14}$$

$$[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0, \tag{15}$$

(BiHom-Jacobi condition)

for all $x, y, z \in L$. The maps α and β (in this order) are called the structure maps of L.

Definition 4.2

A BiHom-Poisson algebra is a 5-tuple $(A, \mu, [\cdot, \cdot], \alpha, \beta)$, with the property that
(1) (A, μ, α, β) is a BiHom-commutative algebra;
(2) $(A, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra;
(3) the following BiHom-Leibniz identity holds for all $x, y, z \in A$:

$$[\alpha\beta(x), yz] = [\beta(x), y]\,\beta(z) + \beta(y)\,[\alpha(x), z]\,. \qquad (16)$$

BiHom-Poisson algebras are defined as a BiHom-type generalization of Poisson algebras (the case $\alpha = \beta = id_A$) and Hom-Poisson algebras (the case $\alpha = \beta$).

A. Makhlouf, S. D. Silvestrov, *Notes on formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. 22 (2010), 715–759.

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Question: under what conditions one can construct a BiHom-Poisson algebra from a BiHom-Novikov-Poisson algebra by taking the commutator bracket of the BiHom-Novikov product?

Definition 4.3 Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra. Then A is called left BiHom-associative if the following condition holds for all $x, y, z \in A$:

$$\alpha(x) * (yz) = (xy) * \beta(z). \tag{17}$$

Definition 4.4 Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra with α, β bijective. Then A is called admissible if $A^{-} := (A, \mu, [\cdot,\cdot], \alpha, \beta)$ is a BiHom-Poisson algebra, where

$$[x, y] = x * y - \alpha^{-1}\beta(y) * \alpha\beta^{-1}(x), \quad \forall\, x, y \in A.$$

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$$[x, y] = x * y - \alpha^{-1}\beta(y) * \alpha\beta^{-1}(x), \quad \forall\, x, y \in A.$$

Theorem 4.5

Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra with α, β bijective. Then A is admissible if and only if it is left BiHom-associative.

Example 4.6

Let (A, μ) be a commutative and associative algebra, $\alpha, \beta : A \to A$ two commuting algebra morphisms, and $D : A \to A$ a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. By Corollary 2.9, $A_{\alpha,\beta} = (A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where $x \bullet y = \alpha(x)\beta(y)$ and $x * y = \alpha(x)D(\beta(y))$, for all $x, y \in A$. Then $A_{\alpha,\beta}$ is left BiHom-associative if and only if $\alpha^2(x)D(\alpha\beta(y))\beta^2(z) = 0$, for all $x, y, z \in A$.

Theorem 4.5

Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra with α, β bijective. Then A is admissible if and only if it is left BiHom-associative.

Example 4.6

Let (A, μ) be a commutative and associative algebra, $\alpha, \beta : A \to A$ two commuting algebra morphisms, and $D : A \to A$ a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. By Corollary 2.9, $A_{\alpha,\beta} = (A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where $x \bullet y = \alpha(x)\beta(y)$ and $x * y = \alpha(x)D(\beta(y))$, for all $x, y \in A$. Then $A_{\alpha,\beta}$ is left BiHom-associative if and only if $\alpha^2(x)D(\alpha\beta(y))\beta^2(z) = 0$, for all $x, y, z \in A$.

The left BiHom-associativity is preserved by perturbations as in Theorem 3.2.

Proposition 4.7 Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then the BiHom-Novikov-Poisson algebra $A' = (A, \diamond, *_{\alpha,\beta}, \alpha^2, \beta^2)$ constructed in Theorem 3.2 is also left BiHom-associative.

In the context of Proposition 4.7 and assuming moreover that α and β are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra $(A')^-$ is given by

$$\begin{aligned} & x *_{\alpha,\beta} y - \alpha^{-2}\beta^2(y) *_{\alpha,\beta} \alpha^2\beta^{-2}(x) \\ = \ & \alpha(x) * \beta(y) - \alpha^{-1}\beta^2(y) * \alpha^2\beta^{-1}(x) \\ = \ & \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) \\ = \ & [\alpha(x), \beta(y)] = [\cdot,\cdot] \circ (\alpha \otimes \beta)(x \otimes y), \end{aligned}$$

where $[\cdot,\cdot]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A^-.

Proposition 3.8 Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then the BiHom-Novikov-Poisson algebra $\overline{A} = (A, \cdot_{\alpha,\beta}, \times, \alpha^2, \beta^2)$ constructed in Theorem 3.3 is also left BiHom-associative.

In the context of Proposition 3.8 and assuming moreover that α and β are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra $(\overline{A})^-$ is given by

$$\begin{aligned}
& x \times y - \alpha^{-2}\beta^2(y) \times \alpha^2\beta^{-2}(x) \\
= \; & \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y) - \alpha^{-1}\beta^2(y) * \alpha^2\beta^{-1}(x) \\
& -\alpha^{-1}\beta^2(y)(\alpha(a)\alpha^2\beta^{-2}(x)) \\
= \; & \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) \\
= \; & [\alpha(x), \beta(y)] = [\cdot,\cdot] \circ (\alpha \otimes \beta)(x \otimes y),
\end{aligned}$$

where $[\cdot,\cdot]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A^-.

From the experience we gained so far in working with BiHom structures, the presence of two maps instead of just one has the following possible types of effects if one tries to extend a classical result to the BiHom case:

(i) it works smoothly, only the proof becomes slightly more complicated than in the Hom case;

(ii) it works, but a new insight is necessary comparing to the Hom case (Lemma 3.1);

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From the experience we gained so far in working with BiHom structures, the presence of two maps instead of just one has the following possible types of effects if one tries to extend a classical result to the BiHom case:

(i) it works smoothly, only the proof becomes slightly more complicated than in the Hom case;

(ii) it works, but a new insight is necessary comparing to the Hom case (Lemma 3.1);

(iii) it does not work (see an example in L. Liu, A. Makhlouf, C. Menini, F. Panaite, *BiHom-pre-Lie algebras, BiHom-Leibniz algebras and Rota-Baxter operators on BiHom-Lie algebras*, Georgian Math. J. 28(2021): 581–594).

It is a classical result that, if $(L, [\cdot, \cdot])$ is a Lie algebra and $R : L \to L$ is a Rota-Baxter operator of weight 0, and one defines a new operation on L by $x \cdot y = [R(x), y]$, then $(L, \cdot)$ is a left pre-Lie algebra.

We wanted to obtain a BiHom analogue (that is, we started with a BiHom-Lie algebra $(L, [\cdot, \cdot], \alpha, \beta)$ and $R : L \to L$ a Rota-Baxter operator of weight 0 such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$, and we wanted to see if $(L, \cdot, \alpha, \beta)$ is a left BiHom-pre-Lie algebra, where $x \cdot y = [R(x), y]$), but it did not work.

Thank you!