Calabi-Yau property under Morita-Takeuchi equivalence

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Outline

- Motivation
- Preliminaries
- Main results

Motivation

Question 1

- *Find homological invariants under monoidally Morita-Tekeuchi equivalence.*

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- *Find homological invariants under monoidally Morita-Tekeuchi equivalence.*
- *Under which conditions, is (twisted) Calabi-Yau property preserved under monoidally Morita-Tekeuchi equivalence? That is, let H and L be two Hopf algebras such that their comodule categories are monoidally equivalent. If H is a (twisted) CY Hopf algebra, when will L still be a (twisted) CY Hopf algebra?*

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  - There is an automorphism μ of A such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^{\mu}, & i = d \end{cases}$$

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- If such an automorphism μ exists, it is unique up to an inner automorphism and is called the Nakayama automorphism of A.
- A CY algebra is a twisted CY algebra whose Nakayama automorphism is an inner automorphism.

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- Let A be an AS-Gorenstein algebra of injective dimension d. Then $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Any non-zero element in $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$ is called a left homological integral of A. We write $\int_A^l$ for $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$.

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  Similarly, we have the right homological integrals. $\mathrm{Ext}_A^d(\mathbb{k}_A, A_A)$ is denoted by $\int_A^r$.
- Let H be a twisted CY Hopf algebra with homological integral $\int_H^l = \mathbb{k}_\xi$, where $\xi : H \to \mathbb{k}$ is an algebra homomorphism. Then the Nakayama automorphism μ of H satisfies $\mu(h) = \xi(h_1)S^2(h_2)$ for any $h \in H$.
  If the right homological integral of H is $\int_H^r = {}_\eta\mathbb{k}$, then $\eta = \xi \circ S$.

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- $\lambda = (\lambda_{ij})_{1\leqslant i<j\leqslant\theta}$: a family of linking parameters for $\mathcal{D}$.
- $V = (x_i, g_i, \chi_i)_{1\leqslant i\leqslant\theta}$: A Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$.

Example1

- The algebra $U(\mathcal{D},\lambda)$ is defined to be the quotient Hopf algebra of the smash product $\Bbbk\langle x_1,\cdots,x_\theta\rangle\#\Bbbk\Gamma$ modulo the ideal generated by the following relations

$$\begin{aligned} &(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j)=0, && 1\leqslant i,j\leqslant\theta,\quad i\neq j,\quad i\sim j,\\ &x_ix_j-\chi_j(g_i)x_jx_i=\lambda_{ij}(1-g_ig_j), && 1\leqslant i<j\leqslant\theta,\quad i\nsim j, \end{aligned}$$

where ad_c is the braided adjoint representation.

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- The pointed Hopf algebras $U(\mathcal{D}, \lambda)$ are generalizations of the quantized enveloping algebras $U_q(\mathfrak{g})$, where $\mathfrak{g}$ are finite dimensional semisimple Lie algebras.

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- Both $U(\mathcal{D}, 0)$ and $U(\mathcal{D}, \lambda)$ are twisted CY Hopf algebras.

Example2

Quantum automorphism groups of non-degenerate bilinear forms [Dubois-Violette and Launer]

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- $E \in \mathrm{GL}_m(\Bbbk)$ with $m \geqslant 2$.
- $\mathcal{B}(E)$ is the algebra presented by generators $(u_{ij})_{1 \leqslant i,j \leqslant m}$ and relations

$$E^{-1}u^tEu = I_m = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1 \leqslant i,j \leqslant m}$, u^t is the transpose of u and I_m is the identity matrix.

Example2

- If $\mathrm{tr}(E^{-1}E^t) = \mathrm{tr}(F^{-1}F^t)$, then the comodule categories of $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are equivalent.

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- (Bichon; Walton, X. Wang) The algebras $\mathcal{B}(E)$ are twisted CY.

- (Wang-Yu-Zhang) Let H be a twisted CY Hopf algebra of dimension d, and L a Hopf algebra monoidally Morita-Takeuchi equivalent to H. If one of the following conditions holds, then L is also twisted CY of dimension d.
  (i) H admits a finitely generated relative projective Yetter-Drinfeld module resolution for the trivial Yetter-Drinfeld module $\mathbb{k}$ and L has finite global dimension.
  (ii) H admits a bounded finitely generated relative projective Yetter-Drinfeld module resolution for the trivial Yetter-Drinfeld module $\mathbb{k}$.
  (iii) H is Noetherian and L has finite global dimension.
  (iv) L is Noetherian and has finite global dimension.

Question 2 (Bichon)

If H and L are two Hopf algebras having equivalent tensor categories of comodules, how are their Hochschild cohomologies related? In particular do H and L have the same cohomological dimension?

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- The cohomological dimension of an algebra A is defined to be $\mathrm{cd}(A) = \sup\{n : \mathrm{H}^n(A, M) \neq 0 \text{ for some } A\text{-bimodule } M\}$

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- (Tailefer) $\mathrm{H}^*_{GS}(H, V) \cong \mathrm{Ext}^*_{\mathcal{YD}^H_H}(\Bbbk, V)$
- $\mathrm{cd}_{GS}(H) = \sup\{n : \mathrm{H}^n_{GS}(H, V) \neq 0 \text{ for some } V \in \mathcal{YD}^H_H\}.$

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Theorem 3 (Bichon)

Let H and L be cosemisimple Hopf algebras having equivalent tensor categories of comodules, and if the antipodes of H and L both satisfy $S^4 = \mathrm{id}$, then $\mathrm{cd}(H) = \mathrm{cd}(L)$.

- (Bichon) Let H and L be two Hopf algebras having equivalent tensor categories of comodules. Then $\mathrm{cd}(H) = \mathrm{cd}(L)$ in the following situations.
  - H and L have bijective antipodes and homologically smooth.
  - H and L are cosemisimple and $\mathrm{cd}(H)$, $\mathrm{cd}(L)$ are finite.
  - H and L are finite dimensional, and the characteristic of $\Bbbk$ is zero, or satisfies $p > d^{\frac{\varphi(d)}{2}}$, where $d = \dim(A)$.
  - H and L are finite dimensional and H^* is unimodular.

A class of cosemisimple Hopf algebras

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- $n \in \mathbb{N}$, $n \geqslant 2$ and let $A, B \in \mathrm{GL}_n(\mathbb{k})$. $\mathcal{G}(A, B)$ is the algebra presented by generators $(u_{ij})_{1 \leqslant i,j \leqslant n}$, $\mathscr{d}$, $\mathscr{d}^{-1}$ and relations

$$u^t A u = A\mathscr{d} \quad uBu^t = B\mathscr{d} \quad \mathscr{d}\mathscr{d}^{-1} = 1 = \mathscr{d}^{-1}\mathscr{d},$$

where u is the matrix $(u_{ij})_{1 \leqslant i,j \leqslant n}$.

- The algebra $\mathcal{G}(A, B)$ admits a Hopf algebra structure, with comultiplication Δ defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, 1 \leqslant i, j \leqslant n, \Delta(\mathscr{d}^{\pm}) = \mathscr{d}^{\pm},$$

with counit defined by

$$\varepsilon(u_{ij}) = \delta_{ij}, 1 \leqslant i, j \leqslant n, \varepsilon(\mathscr{d}) = 1$$

and with antipode S defined by

$$S(u) = \mathscr{d}^{-1} A^{-1} u^t A, S(\mathscr{d}^{\pm}) = \mathscr{d}^{\mp}.$$

- Let $A_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$, for some $q \in \Bbbk^\times$,
  $\mathcal{G}(A_q, A_q^{-1}) = \mathcal{O}(\mathrm{GL}_q(2))$

- Let $A_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$, for some $q \in \mathbb{k}^{\times}$, $\mathcal{G}(A_q, A_q^{-1}) = \mathcal{O}(\mathrm{GL}_q(2))$
- There is a surjective Hopf algebra morphism

$$\mathcal{G}(A, A^{-1}) \to \mathcal{B}(A)$$

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- For any $X, Y \in \mathrm{ob}(\mathcal{C})$, an algebra $\mathcal{C}(X, Y)$.
- For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, algebra homomorphisms

$$\Delta_{XY}^{Z} : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \text{ and } \varepsilon_X : \mathcal{C}(X, X) \to \Bbbk$$

such that for any $X, Y, Z, T \in \mathrm{ob}(\mathcal{C})$, the following diagrams commute:

Cocategories

$$
\begin{array}{ccc}
\mathcal{C}(X,Y) & \xrightarrow{\Delta^{Z}_{X,Y}} & \mathcal{C}(X,Z)\otimes\mathcal{C}(Z,Y) \\
\Delta^{T}_{X,Y}\big\downarrow & & \big\downarrow \Delta^{T}_{X,Z}\otimes 1 \\
\mathcal{C}(X,T)\otimes\mathcal{C}(T,Y) & \xrightarrow{1\otimes\Delta^{Z}_{T,Y}} & \mathcal{C}(X,T)\otimes\mathcal{C}(T,Z)\otimes\mathcal{C}(Z,Y)
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{C}(X,Y) & & \\
\Delta^{Y}_{X,Y}\big\downarrow & \searrow^{=} & \\
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- Thus a cocategory with one object is just a bialgebra.
- A cocategory $\mathcal{C}$ is said to be connected if $\mathcal{C}(X,Y)$ is a non zero algebra for any $X,Y\in \text{ob}(\mathcal{C})$.

Cogroupoids

A cogroupoid $\mathcal{C}$ consists of a cocategory $\mathcal{C}$ together with, for any $X, Y \in \mathrm{ob}(\mathcal{C})$, linear maps

$$S_{X,Y} : \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(Y,X)$$

such that for any $X, Y \in \mathcal{C}$, the following diagrams commute:

$$\begin{array}{ccccc}
\mathcal{C}(X,X) & \xrightarrow{\varepsilon_X} & \mathbb{k} & \xrightarrow{u} & \mathcal{C}(X,Y) \\
\downarrow{\scriptstyle \Delta^Y_{X,X}} & & & & \uparrow{\scriptstyle m} \\
\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X) & & \xrightarrow{1 \otimes S_{Y,X}} & & \mathcal{C}(X,Y) \otimes \mathcal{C}(X,Y)
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- If $\mathcal{C}$ is cogroupoid, then $\mathcal{C}(X, X)$ is a Hopf algebra for any $X \in \mathrm{ob}(\mathcal{C})$.
- We use Sweedler's notation for cogroupoids. Let $\mathcal{C}$ be a cogroupoid. For $a^{X,Y} \in \mathcal{C}(X, Y)$, we write

$$\Delta^{Z}_{X,Y}(a^{X,Y}) = a_1^{X,Z} \otimes a_2^{Z,Y}.$$

Cogroupoids

Theorem 4 (Schauenburg, Bichon)

Let H and L be some Hopf algebras. The following assertions are equivalent.

(1) *There exists a $\Bbbk$-linear equivalence of monoidal categories*

$$\mathrm{Comod}(H) \cong^{\otimes} \mathrm{Comod}(L).$$

(2) *There exists a connected cogroupoid $\mathcal{C}$ with two objects X, Y such that $H = \mathcal{C}(X, X)$ and $L = \mathcal{C}(Y, Y)$.*

Cogroupoids

Theorem 5 (Bichon)

Let $\mathcal{C}$ be a connected cogroupoid. Then for any $X, Y \in \mathrm{ob}(\mathcal{C})$ we have $\Bbbk$-linear equivalences of monoidal categories that are inverse of each other

$$\begin{aligned} \mathrm{Comod}(\mathcal{C}(X,X)) &\cong^{\otimes} \mathrm{Comod}(\mathcal{C}(Y,Y)) \\ V &\mapsto V\Box_{\mathcal{C}(X,X)}\mathcal{C}(X,Y). \end{aligned}$$

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Cogroupoids

Theorem 6 (Bichon)

Let $\mathcal{C}$ be a connected cogroupoid. Then for any $X, Y \in \mathrm{ob}(\mathcal{C})$, the functor

$$\begin{array}{rcl} \mathcal{YD}^{\mathcal{C}(X,X)}_{\mathcal{C}(X,X)} & \to & \mathcal{YD}^{\mathcal{C}(Y,Y)}_{\mathcal{C}(Y,Y)} \\ V & \mapsto & V \square_{\mathcal{C}(X,X)} \mathcal{C}(X,Y). \end{array}$$

is an equivalence of $\mathbb{k}$-linear monoidal categories.

Free YD-module

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- Let V be a right H-comodule. $V \otimes H$ is a Yetter-Drinfeld module with
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- A Yetter–Drinfeld module is said to be relative projective if it is a direct summand of a free Yetter–Drinfeld module.

Free YD-module

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Free YD-module

- Let M be a Yetter-Drinfeld module.
- A free (resp. relative projective) Yetter-Drinfeld resolution of M is a complex of free (resp. relative projective) Yetter-Drinfeld modules

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0$$

for which there exists a Yetter-Drinfeld module morphism $P_0 \to M$ such that

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

is an exact sequence.

The cogroupoid $\mathcal{G}$

The cogroupoid $\mathcal{G}$ is the cogroupoid defined as follows:

- $ob(\mathcal{G}) = \{(A, B) \in \mathrm{GL}_m(\Bbbk) \times \mathrm{GL}_n(\Bbbk), m \geqslant 1\}$,
- For $(A, B), (C, D) \in ob(\mathcal{G})$, the algebra $\mathcal{G}(A, B|C, D)$ is the algebra defined as follows:
  Let $n, m \in \mathbb{N}$, $n, m \geqslant 2$ and let
  $A, B \in \mathrm{GL}_n(\Bbbk), C, D \in \mathrm{GL}_m(\Bbbk)$. The algebra $\mathcal{G}(A, B|C, D)$ is defined to be the algebra with generators u_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, $\mathscr{d}$ and $\mathscr{d}^{-1}$, subject to the relations:

$$u^t A u = C\mathscr{d} \quad u D u^t = B\mathscr{d} \quad \mathscr{d}\mathscr{d}^{-1} = 1 = \mathscr{d}^{-1}\mathscr{d},$$

It is clear that $\mathcal{G}(A, B|A, B) = \mathcal{G}(A, B)$.

- The structure maps are defined as follows: For any $A, B \in \mathrm{GL}_n(\Bbbk)$, $C, D \in \mathrm{GL}_m(\Bbbk)$ and $G \in \mathrm{GL}_p(\mathbb{C})$, define the following maps:

$$\begin{array}{rcl} \Delta^{XY}_{AB,CD} : \mathcal{G}(AB, CD) & \longrightarrow & \mathcal{G}(A, B|X, Y) \otimes \mathcal{G}(X, Y|C, D) \\ u_{ij} & \longmapsto & \sum_{k=1}^{p} u_{ik} \otimes u_{kj} \\ \mathcal{d}^{\pm} & \longmapsto & \mathcal{d}^{\pm} \otimes \mathcal{d}^{\pm}, \end{array}$$

$$\begin{array}{rcl} \varepsilon_{AB} : \mathcal{G}(A, B) & \longrightarrow & \Bbbk \\ u_{ij} & \longmapsto & \delta_{ij} \\ \mathcal{d}^{\pm} & \longmapsto & 1, \end{array}$$

$$\begin{array}{rcl} S_{AB,CD} : \mathcal{G}(AB, CD) & \longrightarrow & \mathcal{G}(CD, AB)^{op} \\ u & \longmapsto & \mathcal{d}^{-1} A^{-1} u^t C \\ \mathcal{d}^{\pm} & \longmapsto & \mathcal{d}^{\mp}. \end{array}$$

It is clear that $S_{AB,CD}$ is bijective.

Lemma 7 (Mrozinski)

Let $\lambda, \mu \in \mathbb{k}^\times$. Consider the full subcogroupoid $\mathcal{G}^{\lambda,\mu}$ of $\mathcal{G}$ with objects

$$ob(\mathcal{G}^{\lambda,\mu}) = \{(A, B) \in ob(\mathcal{G}), B^t A^t BA = \lambda I_n \text{ and } tr(AB^t) = \mu\}$$

Then $ob(\mathcal{G}^{\lambda,\mu})$ is a connected cogroupoid.

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Then $ob(\mathcal{G}^{\lambda,\mu})$ is a connected cogroupoid.

Lemma 8 (Mrozinski)

Let $A, B \in \mathrm{GL}_n(\Bbbk)(n \geqslant 2)$ such that $B^t A^t BA = \lambda I_n$ for some $\lambda \in \Bbbk^\times$ and let $q \in \Bbbk^\times$ such that $q^2 - \sqrt{\lambda^{-1}} tr(AB^t)q + 1 = 0$. Then there is a $\Bbbk$-linear equivalence of monoidal categories

$$\mathrm{Comod}(\mathcal{G}(A, B)) \cong \mathrm{Comod}\,\mathcal{O}(\mathrm{GL}_q(2))$$

between the comodule categories of $\mathcal{G}(A, B)$ and $\mathcal{O}(\mathrm{GL}_q(2))$ respectively.

- $\mathcal{O}(\mathrm{GL}_q(2)) \cong \mathcal{O}(\mathrm{SL}_q(2))[z^{\pm 1}]$

- $\mathcal{O}(\mathrm{GL}_q(2)) \cong \mathcal{O}(\mathrm{SL}_q(2))[z^{\pm 1}]$
- $\mathcal{O}(\mathrm{SL}_q(2)) = \mathcal{B}(A_q)$, where $A_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$

Let $n \in \mathbb{N}$, $n \geqslant 2$ and $A, B \in \mathrm{GL}_n(\Bbbk)$. We define an n-dimensional $\mathcal{G}(A, B)$-comodule $V_{A,B}$ as follows.
It has a basis $v_1^{A,B}, \cdots, v_n^{A,B}$. The comodule action $\rho : V_{A,B} \to V_{A,B} \otimes \mathcal{G}(A, B)$ is defined by

$$\rho(v_i^{A,B}) = \sum_{k=1}^{n} v_k^{A,B} \otimes u_{ki}.$$

The dual vector space $V_{A,B}^*$ is a $\mathcal{G}(A, B)$-comodule with the comodule action $\rho : V_{A,B}^* \to V_{A,B}^* \otimes \mathcal{G}(A, B)$ is defined by

$$\rho(v_i^{A,B*}) = \sum_{j=1}^{n} v_j^{A,B*} \otimes S(u_{ij}).$$

Proposition 9 (Wang-Yu)

Let $\mathcal{A} = \mathcal{O}(\mathrm{GL}_q(2)) = \mathcal{G}(A_q, A_q^{-1})$ *and let* $V = W = V_{A_q, A_q^{-1}}$*. There exists a free Yetter-Drinfeld resolution of the counit of the algebra* $\mathcal{A}$

$$0 \to P_4 \xrightarrow{\phi_4} P_3 \xrightarrow{\phi_3} P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\varepsilon} \mathbb{k} \to 0$$

where $P_4 = \mathbb{k} \boxtimes \mathcal{A}$, $P_3 = (V^* \otimes V) \boxtimes \mathcal{A} \oplus \mathcal{A}$, $P_2 = (V^* \otimes V) \boxtimes \mathcal{A} \oplus (W^* \otimes W) \boxtimes \mathcal{A}$, $P_1 = (W^* \otimes W) \boxtimes \mathcal{A} \oplus \mathcal{A}$ *and* $P_0 = \mathbb{k} \boxtimes \mathcal{A}$.

We denote the canonical basis of V and W by v_1, v_2 and w_1, w_2 respectively. The morphisms $\phi_1, \phi_2, \phi_3, \phi_4$ are defined as follows. $\phi_4(x) = v_1^* \otimes v_1 \otimes ((-q + q^{-1}d)x) + v_1^* \otimes v_2 \otimes (-cx) + v_2^* \otimes v_1 \otimes (-bx) + v_2^* \otimes v_2 \otimes ((-q^{-1} + qa)x) + dx$

$$\phi_3(v_1^* \otimes v_1 \otimes x) =$$
$$v_1^* \otimes v_1 \otimes x + v_2^* \otimes v_1 \otimes (-q^{-1}bx) + v_2^* \otimes v_2 \otimes ax + w_1^* \otimes w_1 \otimes \mathscr{d}x$$
$$\phi_3(v_1^* \otimes v_2 \otimes x) = v_1^* \otimes v_1 \otimes bx + v_1^* \otimes v_2 \otimes (1 - qa)x + w_1^* \otimes w_2 \otimes \mathscr{d}x$$
$$\phi_3(v_2^* \otimes v_1 \otimes x) = v_2^* \otimes v_1 \otimes (1 - q^{-1}d)x + v_2^* \otimes v_2 \otimes cx + w_2^* \otimes w_1 \otimes \mathscr{d}x$$
$$\phi_3(v_2^* \otimes v_2 \otimes x) =$$
$$v_2^* \otimes v_2 \otimes x + v_1^* \otimes v_1 \otimes dx + v_1^* \otimes v_2 \otimes (-qc)x + w_2^* \otimes w_2 \otimes \mathscr{d}x$$
$$\phi_3(x) =$$
$$-(w_1^* \otimes w_1 \otimes ((-q + q^{-1}d)x) + w_1^* \otimes w_2 \otimes (-cx) + w_2^* \otimes w_1 \otimes (-bx) + w_2^* \otimes w_2 \otimes ((-q^{-1} + qa)x) + dx) - v_1^* \otimes v_1 \otimes q(1 - \mathscr{d}^{-1}) - v_1^* \otimes v_1 \otimes q^{-1}(1 - \mathscr{d}^{-1})$$
$$\phi_2(v_1^* \otimes v_1 \otimes x) = (a-1)x + w_1^* \otimes w_1 \otimes \mathscr{d}x$$
$$\phi_2(v_1^* \otimes v_2 \otimes x) = bx + w_1^* \otimes w_2 \otimes \mathscr{d}x$$
$$\phi_2(v_2^* \otimes v_1 \otimes x) = cx + w_2^* \otimes w_1 \otimes \mathscr{d}x$$
$$\phi_2(v_2^* \otimes v_2 \otimes x) = (d-1)x + w_2^* \otimes w_2 \otimes \mathscr{d}x$$

$$\phi_2(w_1^* \otimes w_1 \otimes x) =$$
$$-(w_1^* \otimes w_1 \otimes x + w_2^* \otimes w_1 \otimes (-q^{-1}bx) + w_2^* \otimes w_2 \otimes ax) - (1 - \mathscr{d}^{-1})x$$
$$\phi_2(w_1^* \otimes w_2 \otimes x) = -(w_1^* \otimes w_1 \otimes bx + w_1^* \otimes w_2 \otimes (1 - qa)x)$$
$$\phi_2(w_2^* \otimes w_1 \otimes x) = -(w_2^* \otimes w_1 \otimes (1 - q^{-1}d)x + w_2^* \otimes w_2 \otimes cx)$$
$$\phi_2(w_2^* \otimes w_2 \otimes x) =$$
$$-(w_2^* \otimes w_2 \otimes x + w_1^* \otimes w_1 \otimes dx + w_1^* \otimes w_2 \otimes (-qc)x) - (1 - \mathscr{d}^{-1})x$$
$$\phi_1(w_1^* \otimes w_1) = -(a-1) \ \phi_1(w_1^* \otimes w_2) = -b$$
$$\phi_1(w_2^* \otimes w_1) = -c \ \phi_1(w_2^* \otimes w_2) = -(d-1) \ \phi_1(x) = \mathscr{d}x$$

Proposition 10 (Wang-Yu)

The algebras $\mathcal{O}(\mathrm{GL}_q(2))$ are twisted CY algebras with Nakayama automorphism μ defined by
$\mu(a) = q^2 a, \mu(b) = b, \mu(c) = c, \mu(d) = q^{-2} d.$

Theorem 11 (Wang-Yu)

Let $A, B \in \mathrm{GL}_n(\Bbbk)$ $(n \geqslant 2)$ *such that* $B^t A^t BA = \lambda I_n$ *for some* $\lambda \in \Bbbk^*$ *and let* $q \in \Bbbk^\times$ *such that* $q^2 - \sqrt{\lambda^{-1}}tr(AB^t)q + 1 = 0$. *The algebra* $\mathcal{G}(A, B)$ *is a twisted CY algebra.*
Recall that for $u_{ij} \in \mathcal{G}(A_q, A_q^{-1}|A, B)$, $i = 1, 2$, $1 \leqslant j \leqslant n$, $\Delta^{AB}_{A_q A_q^{-1}, AB}(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. *The right homological integral* η *of* $\mathcal{G}(A, B)$ *satisfies the followiing equation:*

$$\sum_{k=1}^{n} u_{ik}((A^t)^{-1}B(A^t)^{-1}B)_{kj} = \sum_{k=1}^{n} \gamma(u_{ik})\eta(u_{kj}),$$

where γ *is an inner automorphism of* $\mathcal{G}(A_q, A_q^{-1}|A, B)$.

Thank you!