Classification of fusion categories and factorizable semisimple Hopf algebras

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2021. 8. 23

Motivations

- M. Müger defined Morita equivalence of fusion categories [J. Pure Appl. Algebra, 2003], and proposed the minimal modular extension conjecture [Proc. Lond. Math. Soc, 2003].
- T. Lan, L. Kong, X. Wen studied the minimal extension conjecture from the aspect of topological orders [Comm. Phys. Math, 2017].
- In [Adv. Math, 2011], P. Etingof, D. Nikshych, V. Ostrik defined weakly group-theoretical fusion categories and solvable fusion categories, which are closely related finite groups, (quasi-)Hopf algebras.

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Let $\mathbb{K}$ be a field, $\mathbb{K} = \overline{\mathbb{K}}$, char$(\mathbb{K}) = 0$.

Definition

$\mathcal{C}$ is a finite tensor category if:

- *$\mathcal{C} \cong Rep(A)$ as a $\mathbb{K}$-linear abelian category, where A is a f-d unital associative $\mathbb{K}$-algebra, $Rep(A)$ is the category of f-d left A-modules.*
- *$\mathcal{C} = (\mathcal{C}, \mathbb{I}, \otimes, \rho, \lambda, \alpha)$ is a monoidal category, and $- \otimes -$ is a $\mathbb{K}$-linear exact bifunctor.*
- *Every object X has a left dual X^* and right dual ${}^*X$.*
- *$\mathbb{I}$ is a simple object.*

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Examples of fusion categories

- Vec, the category of finite-dimensional vector spaces, with tensor product $\otimes_{\mathbb{K}}$ over field $\mathbb{K}$.
- Pointed fusion category Vec_G^{ω}, the category of G-graded finite-dimensional vector spaces category, where G is a finite group, $\omega \in Z^3(G, \mathbb{K}^*)$ is a normalized 3-cocycle, $\mathbb{K}^* := \mathbb{K}\backslash\{0\}$. Explicitly, $\alpha_{V_{g_1}, V_{g_2}, V_{g_3}} = \omega(g_1, g_2, g_3) : (V_{g_1} \otimes V_{g_2}) \otimes V_{g_3} \xrightarrow{\sim} V_{g_1} \otimes (V_{g_2} \otimes V_{g_3})$.
- Let G be a finite group, then $\mathrm{Rep}(G)$ is a fusion category.
- Let H be a f-d (quasi-)Hopf algebra, then $\mathrm{Rep}(H)$ is a finite tensor category. $\mathrm{Rep}(H)$ is a fusion category iff H is semisimple (quasi-)Hopf algebra.

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Frobenius-Perron dimension

Definition (ENO, Ann. of. Math, 2005)

Let $Gr(\mathcal{C})$ be the Grothendieck ring of fusion category $\mathcal{C}$. Then there exists a unique homomorphism $FPdim(-) : Gr(\mathcal{C}) \to \mathbb{K}$ such that $FPdim(X)$ is a positive algebraic integer, $\forall X \in \mathcal{O}(\mathcal{C})$.

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The sum $FPdim(\mathcal{C}) := \sum_{X \in \mathcal{O}(\mathcal{C})} FPdim(X)^2$ is defined as the Frobenius-Perron dimension of $\mathcal{C}$.

Example

$\mathcal{C} = Rep(H)$, where H is a s.s (quasi-)Hopf algebra. Then $FPdim(X) = \dim_{\mathbb{K}}(X)$ for $X \in \mathcal{O}(\mathcal{C})$. In particular,

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Integral fusion categories

Definition

A fusion category $\mathcal{C}$ is integral if FPdim$(X) \in \mathbb{Z}, \forall X \in \mathcal{O}(\mathcal{C})$.

Theorem (ENO, Ann. of. Math, 2005)

A fusion category $\mathcal{C}$ is integral iff $\mathcal{C} \cong Rep(H)$ for some s.s quasi-Hopf algebra H.

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A fusion category $\mathcal{C}$ is weakly integral if FPdim$(\mathcal{C}) \in \mathbb{Z}$.

Example

Let $d \in \mathbb{Z}_{\geq 3}$ be square-free. Then Tambara-Yamagami fusion category $\mathcal{C} := \mathcal{TY}(\mathbb{Z}_d, \chi, \nu)$ is weakly integral: $\mathcal{O}(\mathcal{C}) = \{X\} \cup \mathbb{Z}_d$,

$$X \otimes X = \oplus_{g \in \mathbb{Z}_d} g, \; g \otimes X = X \otimes g = X.$$

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Let $\mathcal{C}$ be a fusion category. The adjoint subcategory $\mathcal{C}_{ad}$ of $\mathcal{C}$ is the fusion category generated by object $X \in \mathcal{O}(\mathcal{C})$ such that

- $X \subseteq Y \otimes Y^*$ *for some* $Y \in \mathcal{O}(\mathcal{C})$.

Definition (S. Gelaki, D. Nikshych, Adv. Math, 2008)

Fusion category $\mathcal{C}$ is nilpotent, if $\mathcal{C}^{(n)} = Vec$, where $\mathcal{C}^{(0)} := \mathcal{C}$, $\mathcal{C}^{(1)} := \mathcal{C}_{ad}$, $\mathcal{C}^{(m)} := \mathcal{C}_{ad}^{(m-1)}$, $m \in \mathbb{Z}_{\geq 1}$.

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Let G be a finite group, $Z(G)$ the center of G, $\mathcal{C} = Rep(G)$. Then $\mathcal{C}_{ad} = Rep(G/Z(G))$. Thus, $\mathcal{C}$ is nilpotent iff G is nilpotent.

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Weakly group-theoretical fusion categories

Remark (S. Gelaki, D. Nikshych, Adv. Math, 2008)

Indeed, $\mathcal{C}$ is nilpotent iff there exist a series of fusion subcategories $\mathcal{C}_0 = Vec \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n = \mathcal{C}$ such that $\mathcal{C}_j$ is a G_j-graded extension of $\mathcal{C}_{j-1}$, where G_j are finite groups.

Theorem (ENO, Adv. Math, 2011)

Let $\mathcal{C}$ be a fusion category.

- *$\mathcal{C}$ is group-theoretical iff $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(Vec_G^\omega)$.*
- *$\mathcal{C}$ is weakly group-theoretical iff $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$, where $\mathcal{D}$ is a nilpotent fusion category.*

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Are weakly integral fusion categories weakly group-theoretical?

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Indeed, $\mathcal{C}$ is nilpotent iff there exist a series of fusion subcategories $\mathcal{C}_0 = Vec \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n = \mathcal{C}$ such that $\mathcal{C}_j$ is a G_j-graded extension of $\mathcal{C}_{j-1}$, where G_j are finite groups.

Theorem (ENO, Adv. Math, 2011)

Let $\mathcal{C}$ be a fusion category.

- *$\mathcal{C}$ is group-theoretical iff $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(Vec_G^\omega)$.*
- *$\mathcal{C}$ is weakly group-theoretical iff $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$, where $\mathcal{D}$ is a nilpotent fusion category.*

Question (ENO. Adv. Math, 2011)

Are weakly integral fusion categories weakly group-theoretical?

Frobenius property

Theorem (ENO. Adv. Math, 2011)

A weakly group-theoretical fusion category $\mathcal{C}$ admits Frobenius property. That is, $\frac{FPdim(\mathcal{C})}{FPdim(X)}$ is an algebraic integer, $\forall X \in \mathcal{O}(\mathcal{C})$.

Remark

- *If the previous conjecture is true, then the Kaplansky's sixth conjecture is true.*
- *Weakly group-theoretical property is invariant under taking Drinfeld centers, (de-)equivariantizations, G-extensions.*

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Are the Drinfeld centers of weakly integral fusion categories weakly group-theoretical?

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Braided fusion category

Definition

Fusion category $\mathcal{C}$ is a braided fusion category if there exists a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, $X, Y \in \mathcal{C}$ satisfying the braiding equations.

Example

- $\mathcal{C} = Rep(G)$, where the braiding c of $Rep(G)$ is given by the reflection τ of vector spaces. We call $\mathcal{C}$ a Tannakian category.
- $\mathcal{C} = Rep(G, z)$, where $z \in Z(G)$ and $z^2 = 1$, the braiding of $Rep(G, z)$ is given by $\tau \circ R$ with

$$R = \frac{1}{2}(1 \otimes 1 + z \otimes 1 + 1 \otimes z - z \otimes z).$$

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Müger centralizer

Definition (M. Müger, Adv. Math, 2000)

Let $\mathcal{C} \subseteq \mathcal{D}$ be braided fusion categories. The following braided fusion category

$$\mathcal{D}'_{\mathcal{C}} := \{Y \in \mathcal{C} | c_{Y,X}c_{X,Y} = id_{X\otimes Y}, \forall X \in \mathcal{D}\}$$

is called the Müger centralizer of $\mathcal{C}$ in $\mathcal{D}$. Denote the Müger center of $\mathcal{C}$ by $\mathcal{C}' := \mathcal{C}'_{\mathcal{C}}$.

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Let $\mathcal{C}$ be a braided fusion category.

- *$\mathcal{C}$ is non-degenerate if $\mathcal{C}' = Vec$.*
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Example

- *Let $\mathcal{C}$ be a fusion category, then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is non-degenerate.*
- *Let $k = 4m + 2 \in \mathbb{Z}_{\geq 2}$ and $\mathcal{C} := \mathcal{C}(\mathfrak{sl}_2, k + 2, q)$ with $q^2 = e^{\frac{2\pi i}{k+2}}$, then $\mathcal{C}$ is non-degenerate, while $\mathcal{C}_{ad}$ is slightly degenerate.*

Theorem (P. Deligne, Moscow Math. J, 2002)

Any symmetric fusion category $\mathcal{C}$ is braided equivalent to Rep(G, z). In particular, $\mathcal{C}$ contains a maximal Tannakian subcategory Rep$(G/\langle z\rangle)$.

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Let G be a finite abelian group. $\eta : G \to \mathbb{K}^*$ *is a quadratic form on G, if*

- $\eta(g) = \eta(g^{-1}), \forall g \in G$,
- $B(g,h) := \frac{\eta(gh)}{\eta(g)\eta(h)}$ *defines a symmetric bi-character.*

η *is non-degenerate if* $B(-,-)$ *is.*

Theorem (A. Joyal, R. Street, Adv. Math, 1993)

There exists a bijective correspondence between metric groups and pointed non-degenerate fusion categories.

We denote the category determined by metric group (G,η) by $\mathcal{C}(G,\eta)$ below.

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Theorem (V. Ostrik, Z. Yu, arXiv:2105.0181)

Let $n, d \in \mathbb{Z}_{\geq 1}$, $(n, d) = 1$ and d is square-free. Assume that $\mathcal{C}$ is a non-degenerate fusion category with $\text{FPdim}(\mathcal{C}) = nd$. If

- *$\mathcal{C}$ contains a Tannakian category $\mathcal{E} := \text{Rep}(G)$ such that $(\mathcal{E}'_{\mathcal{C}})_G \cong \mathcal{A} \boxtimes \mathcal{C}(\mathbb{Z}_d, \eta)$ as braided fusion category,*
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- *The condition $(FPdim(X)^2, d) = 1$ is necessary. Assume $\mathcal{C} := \mathcal{TY}(\mathbb{Z}_d, \chi, \mu)^{\mathbb{Z}_2}$, then $\mathcal{C}$ contains two simple objects of FP-dimension $\sqrt{d}$, but $\mathcal{C}$ does not contain $\mathcal{C}(\mathbb{Z}_d, \eta)$.*

Corollary (V. Ostrik, Z. Yu, arXiv:2105.0181)

Let $\mathcal{C}$ be an integral non-degenerate fusion category with $FPdim(\mathcal{C}) = nd$, $n, d \in \mathbb{Z}_{\geq 1}$, $(n, d) = 1$ and d is square-free.

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The subcase that $n = p^a$ with p being an odd prime was proved in [J. Dong, S. Natale, Algebr. Represent. Theory, 2018].

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Definition (M. Müger, Proc. Lond. Math. Soc, 2003)

Let $\mathcal{C}$ be a braided fusion category. A non-degenerate fusion category $\mathcal{D}$ is a minimal extension of $\mathcal{C}$ if $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{C}'_{\mathcal{D}} \cong \mathcal{C}'$.

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- *Indeed, $\mathcal{D}$ is a minimal extension of $\mathcal{C}$ if and only if $\mathcal{C} \subseteq \mathcal{D}$ and $FPdim(\mathcal{D}) = FPdim(\mathcal{C})FPdim(\mathcal{C}')$.*
- *However, a braided fusion category may not admit a minimal extension (unpublished note by V. Drinfeld).*

Theorem (V. Ostrik, Z. Yu, arXiv:2105.0181)

Let $\mathcal{C}$ be a slightly degenerate fusion category. If $\mathcal{C}$ is Witt equivalent to a weakly group-theoretical fusion category, then $\mathcal{C}$ admits a minimal extension.

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Factorizable Hopf algebra

Definition

Let (H, R) be a f-d quasi-triangular Hopf algebra over $\mathbb{K}$. H is factorizable if the Drinfeld map $\Phi : H^* \to H$ is an isomorphism, where $\Phi(f) = (id \otimes f)(R_{21}R)$.

Example

- *Drinfeld double $(\mathcal{D}(H), R)$ is factorizable, where H is a f-d Hopf algebra and $R = \sum_i (1 \otimes h^i) \otimes (h_i \otimes \varepsilon)$.*
- *$(\mathbb{K}[\mathbb{Z}_d], R_m)$ is a factorizable Hopf algebra with*

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In fact, a f-d (quasi-)Hopf algebra H is factorizable if and only if $Rep(H)$ is non-degenerate.

Theorem

Let H be a factorizable semisimple Hopf algebra. Assume $\dim_{\mathbb{K}}(H) = nd$ where $(n, d) = 1$ and d is square-free. If $Rep(H)$ is weakly group-theoretical, then $H \cong L \times (\mathbb{K}[\mathbb{Z}_d], R_m)$.

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For factorizable s.s quasi-Hopf algebra H with same properties, we have $H \cong A \times (\mathbb{K}^\omega[\mathbb{Z}_d], R)$, $\omega \in \mathbb{Z}^3(\mathbb{Z}_d, \mathbb{K}^*)$.

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Thank you for your attentions !